Computer Algebra
Independent Integration Tests

Summer 2023 edition with Rubi V 4.17.3

4-Trig-functions/4.1-Sine/75-4.1.2.3-g-sin-^p-a+b-sin-^m-c+d-sin-^n

Nasser M. Abbasi

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Contents

CHAPTER 1

INTRODUCTION

This report gives the result of running the computer algebra independent integration test.The download section in on the main webpage contains links to download the problems in plain text format used for all CAS systems. The number of integrals in this report is [51]. This is test number [75].

1.1 Listing of CAS systems tested

The following are the CAS systems tested:

1. Mathematica 13.3.1 (August 16, 2023) on windows 10.
2. Rubi 4.17.3 (Sept 25, 2023) on Mathematica 13.3.1 on windows 10
3. Maple 2023.1 (July, 12, 2023) on windows 10.
4. Maxima 5.47 (June 1, 2023) using Lisp SBCL 2.3.0 on Linux via sagemath 10.1 (Aug 20, 2023).
5. FriCAS 1.3.9 (July 8, 2023) based on sbcl 2.3.0 on Linux via sagemath 10.1 (Aug 20, 2023).
6. Giac/Xcas 1.9.0-57 (June 26, 2023) on Linux via sagemath 10.1 (Aug 20, 2023).
7. Sympy 1.12 (May 10, 2023) Using Python 3.11.3 on Linux.
8. Mupad using Matlab 2021a with Symbolic Math Toolbox Version 8.7 on windows 10.

Maxima and Fricas and Giac are called using Sagemath. This was done using Sagemath `integrate` command by changing the name of the algorithm to use the different CAS systems.

Sympy was run directly in Python not via sagemath.

1.2 Results

Important note: A number of problems in this test suite have no antiderivative in closed form. This means the antiderivative of these integrals can not be expressed in terms of elementary, special functions or `Hypergeometric2F1` functions. `RootSum` and `RootOf` are not allowed. If a CAS returns the above integral unevaluated within the time limit, then the result is counted as passed and assigned an A grade.

However, if CAS times out, then it is assigned an F grade even if the integral is not integrable, as this implies CAS could not determine that the integral is not integrable in the time limit.

If a CAS returns an antiderivative to such an integral, it is assigned an A grade automatically and this special result is listed in the introduction section of each individual test report to make it easy to identify as this can be important result to investigate.

The results given in in the table below reflects the above.

System	% solved	% Failed
Rubi	100.00 (51)	0.00 (0)
Maple	98.04 (50)	1.96 (1)
Mathematica	92.16 (47)	7.84 (4)
Fricas	60.78 (31)	39.22 (20)
Giac	39.22 (20)	60.78 (31)
Maxima	31.37 (16)	68.63 (35)
Mupad	25.49 (13)	74.51 (38)
Sympy	7.84 (4)	92.16 (47)

Table 1.1: Percentage solved for each CAS

The table below gives additional break down of the grading of quality of the antiderivatives generated by each CAS. The grading is given using the letters A,B,C and F with A being the best quality. The grading is accomplished by comparing the antiderivative generated with the optimal antiderivatives included in the test suite. The following table describes the meaning of these grades.

grade	description
A	Integral was solved and antiderivative is optimal in quality and leaf size.
B	Integral was solved and antiderivative is optimal in quality but leaf size is larger than twice the optimal antiderivatives leaf size.
C	Integral was solved and antiderivative is non-optimal in quality. This can be due to one or more of the following reasons 1. antiderivative contains a hypergeometric function and the optimal antiderivative does not. 2. antiderivative contains a special function and the optimal antiderivative does not. 3. antiderivative contains the imaginary unit and the optimal antiderivative does not.
F	Integral was not solved. Either the integral was returned unevaluated within the time limit, or it timed out, or CAS hanged or crashed or an exception was raised.

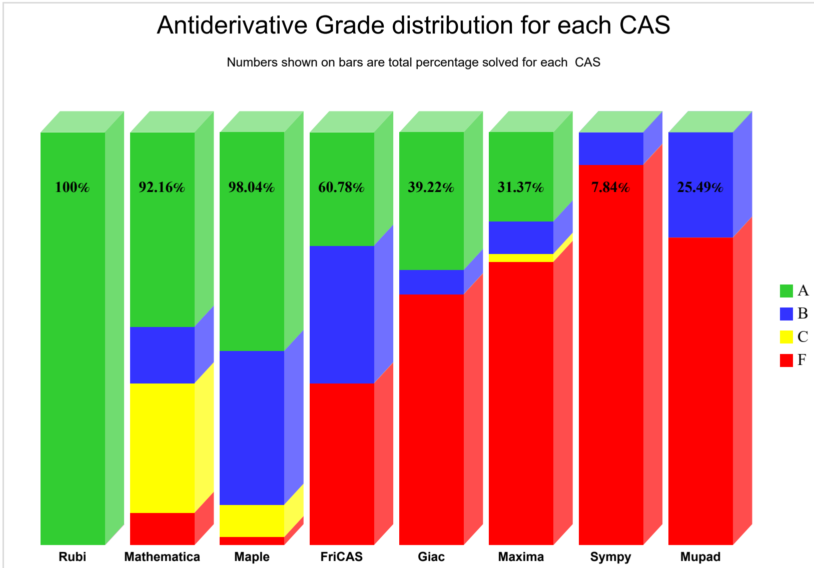
Table 1.2: Description of grading applied to integration result

Grading is implemented for all CAS systems. Based on the above, the following table summarizes the grading for this test suite.

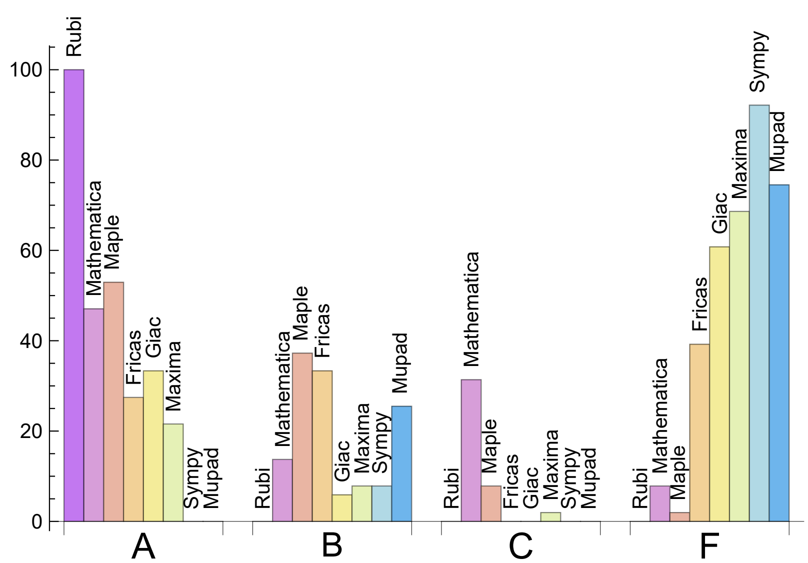
System	% A grade	% B grade	% C grade	% F grade
Rubi	100.000	0.000	0.000	0.000
Maple	52.941	37.255	7.843	1.961
Mathematica	47.059	13.725	31.373	7.843
Giac	33.333	5.882	0.000	60.784
Fricas	27.451	33.333	0.000	39.216
Maxima	21.569	7.843	1.961	68.627
Mupad	0.000	25.490	0.000	74.510
Sympy	0.000	7.843	0.000	92.157

Table 1.3: Antiderivative Grade distribution of each CAS

The following is a Bar chart illustration of the data in the above table.



The figure below compares the grades of the CAS systems.



The following table shows the distribution of the different types of failures for each CAS. There are 3 types failures. The first is when CAS returns the input within the time limit, which means it could not solve it. This is the typical failure and given as **F**.

The second failure is due to time out. CAS could not solve the integral within the 3 minutes time limit which is assigned. This is assigned **F(-1)**.

The third is due to an exception generated, indicated as **F(-2)**. This most likely indicates an interface problem between sagemath and the CAS (applicable only to FriCAS, Maxima and Giac) or it could be an indication of an internal error in the CAS itself. This type of error requires more investigation to determine the cause.

System	Number failed	Percentage normal failure	Percentage timeout failure	Percentage exception failure
Rubi	0	0.00	0.00	0.00
Maple	1	100.00	0.00	0.00
Mathematica	4	100.00	0.00	0.00
Fricas	20	40.00	60.00	0.00
Giac	31	67.74	25.81	6.45
Maxima	35	97.14	0.00	2.86
Mupad	38	0.00	100.00	0.00
Sympy	47	89.36	10.64	0.00

Table 1.4: Failure statistics for each CAS

1.3 Time and leaf size Performance

The table below summarizes the performance of each CAS system in terms of time used and leaf size of results.

Mean size is the average leaf size produced by the CAS (before any normalization). The Normalized mean is relative to the mean size of the optimal anti-derivative given in the input files.

For example, if CAS has **Normalized mean** of 3, then the mean size of its leaf size is 3 times as large as the mean size of the optimal leaf size.

Median size is value of leaf size where half the values are larger than this and half are smaller (before any normalization). i.e. The Middle value.

Similarly the **Normalized median** is relative to the median leaf size of the optimal.

For example, if a CAS has Normalized median of 1.2, then its median is 1.2 as large as the median leaf size of the optimal.

System	Mean time (sec)
Sympy	0.24
Maxima	0.24
Giac	0.37
Rubi	0.74
Maple	2.14
Fricas	2.62
Mathematica	7.78
Mupad	13.99

Table 1.5: Time performance for each CAS

System	Mean size	Normalized mean	Median size	Normalized median
Giac	124.85	1.35	114.50	1.30
Maxima	142.81	1.98	121.50	1.56
Rubi	148.98	1.02	123.00	1.00
Sympy	273.50	3.08	273.00	3.16
Fricas	952.00	7.60	240.00	3.31
Mathematica	1794.38	8.26	203.00	1.57
Mupad	2002.92	12.16	163.00	2.21
Maple	7015.90	22.79	225.00	1.86

Table 1.6: Leaf size performance for each CAS

1.4 Performance based on number of rules Rubi used

This section shows how each CAS performed based on the number of rules Rubi needed to solve the same integral. One diagram is given for each CAS.

On the y axis is the percentage solved which Rubi itself needed the number of rules given the x axis. These plots show that as more rules are needed then most CAS system percentage of solving decreases which indicates the integral is becoming more complicated to solve.

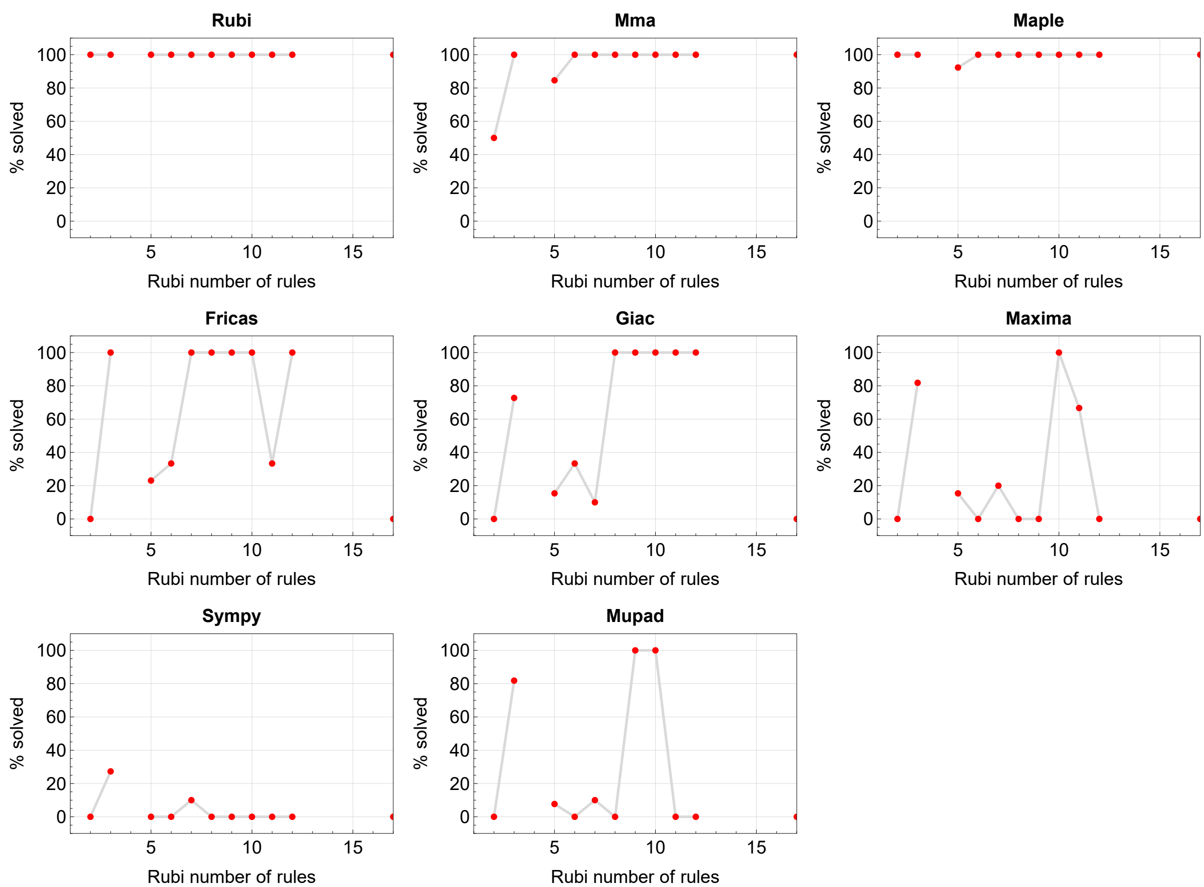


Figure 1.1: Solving statistics per number of Rubi rules used

1.5 Performance based on number of steps Rubi used

This section shows how each CAS performed based on the number of steps Rubi needed to solve the same integral. Note that the number of steps Rubi needed can be much higher than the number of rules, as the same rule could be used more than once.

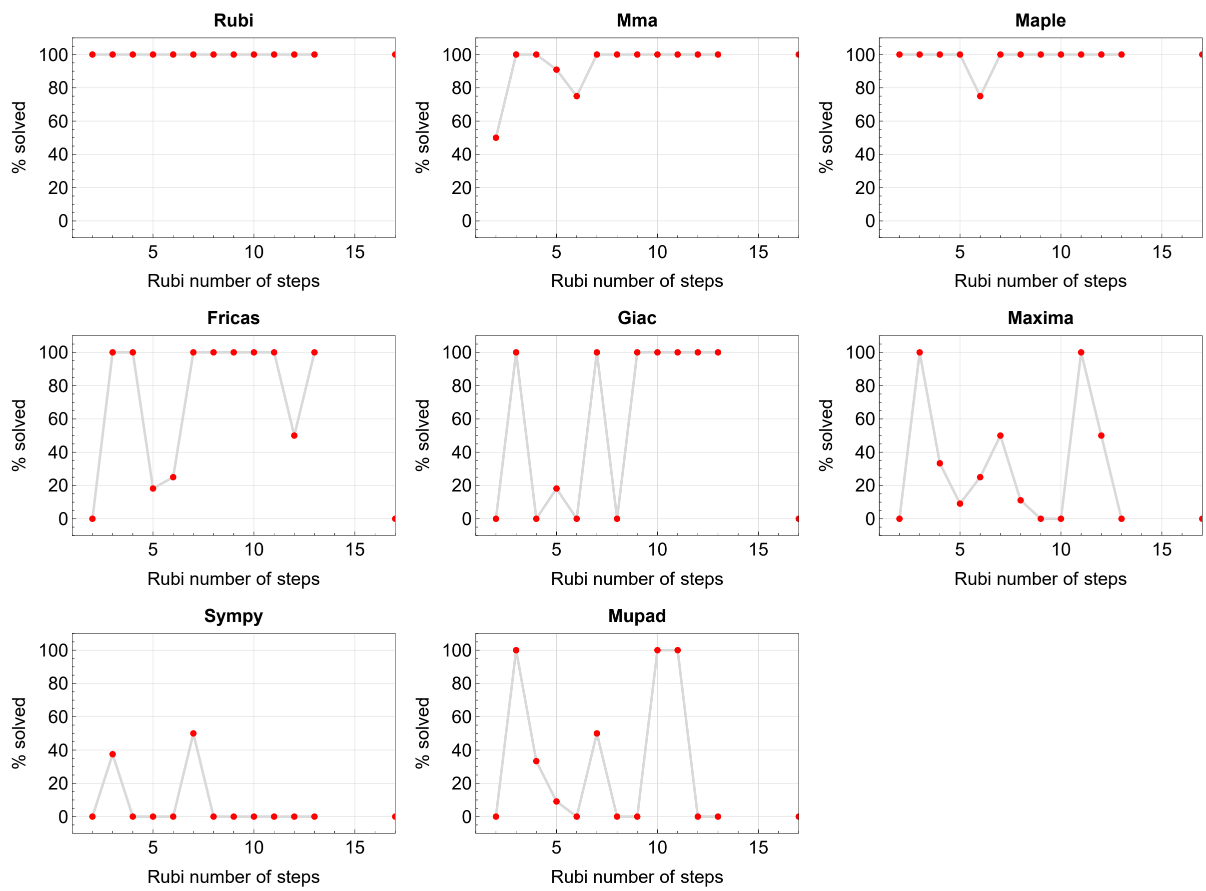


Figure 1.2: Solving statistics per number of Rubi steps used

The above diagram show that the precentage of solved intergals decreases for most CAS systems as the number of steps increases. As expected, for integrals that required less steps by Rubi, CAS systems had more success which indicates the integral was not as hard to solve. As Rubi needed more steps to solve the integral, the solved percentage decreased for most CAS systems which indicates the integral is becoming harder to solve.

1.6 Solved integrals histogram based on leaf size of result

The following shows the distribution of solved integrals for each CAS system based on leaf size of the antiderivatives produced by each CAS. It shows that most integrals solved produced leaf size less than about 100 to 150. The bin size used is 40.

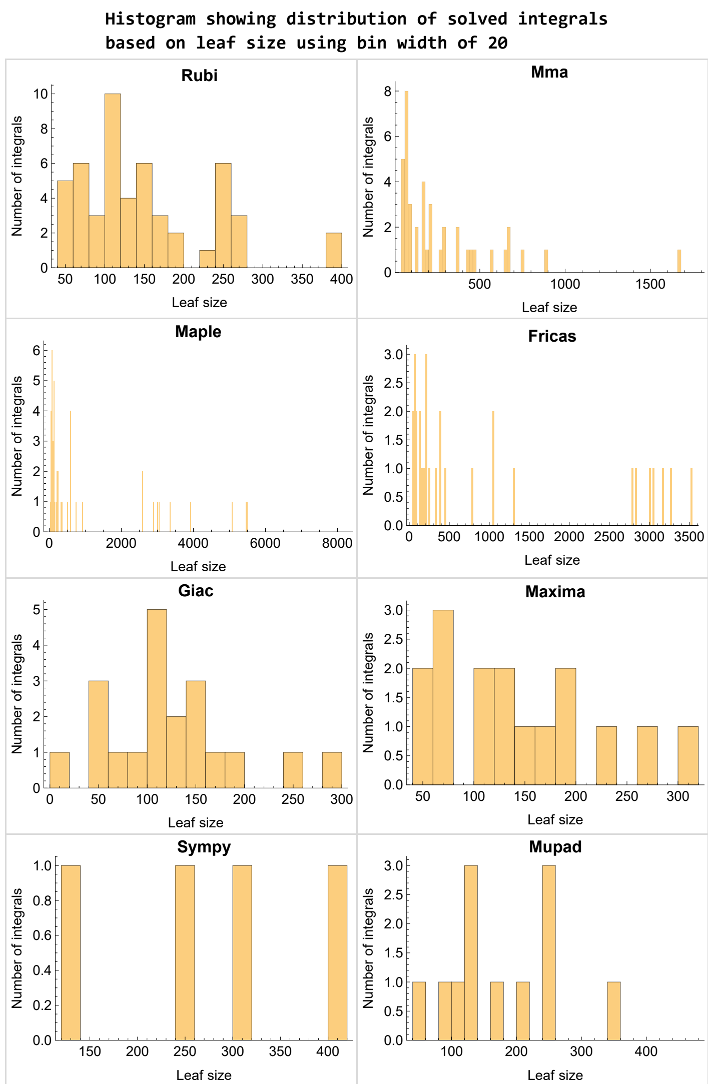


Figure 1.3: Solved integrals based on leaf size distribution

1.7 Solved integrals histogram based on CPU time used

The following shows the distribution of solved integrals for each CAS system based on CPU time used in seconds. The bin size used is 0.1 second.

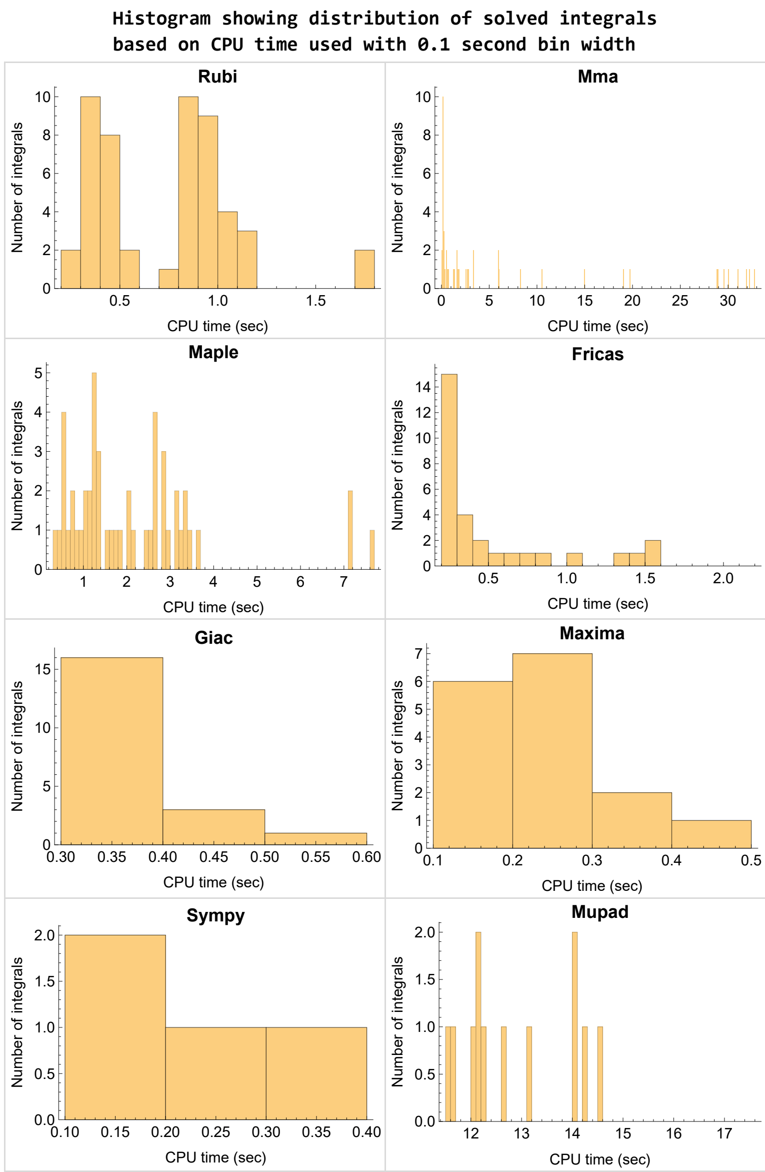


Figure 1.4: Solved integrals histogram based on CPU time used

1.8 Leaf size vs. CPU time used

The following gives the relation between the CPU time used to solve an integral and the leaf size of the antiderivative.

The result for Fricas, Maxima and Giac is shifted more to the right than the other CAS system due to the use of sagemath to call them, which causes an initial slight delay in the timing to start the integration due to overhead of starting a new process each time.

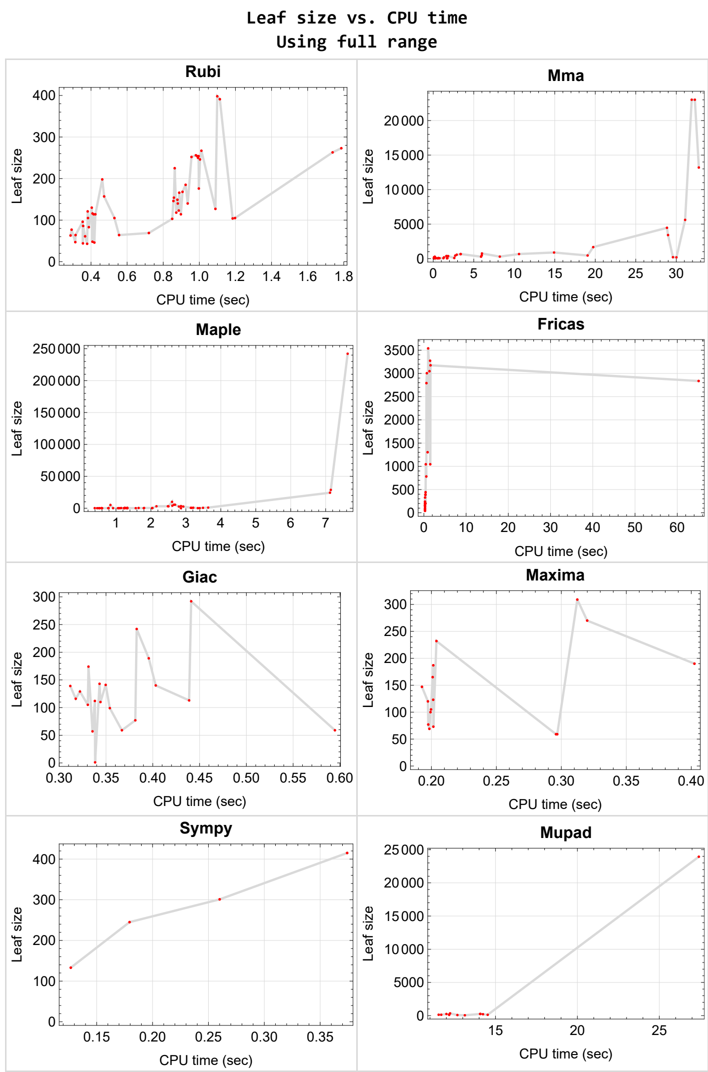


Figure 1.5: Leaf size vs. CPU time. Full range

1.9 list of integrals with no known antiderivative

{}

1.10 List of integrals solved by CAS but has no known antiderivative

Rubi {}

Mathematica {}

Maple {}

Maxima {}

Fricas {}

Sympy {}

Giac {}

Mupad {}

1.11 list of integrals solved by CAS but failed verification

The following are integrals solved by CAS but the verification phase failed to verify the anti-derivative produced is correct. This does not necessarily mean that the anti-derivative is wrong as additional methods of verification might be needed, or more time is needed (3 minutes time limit was used). These integrals are listed here to make it possible to do further investigation to determine why the result could not be verified.

Rubi {}

Mathematica $\{26, 31, 32, 33, 34, 35, 36, 37, 38, 42, 45, 46\}$

Maple $\{15, 25, 26, 27, 28, 31, 32, 33, 34, 35, 37, 38, 42, 43, 44, 45, 46, 47, 48, 49, 50\}$

Maxima Verification phase not currently implemented.

Fricas Verification phase not currently implemented.

Sympy Verification phase not currently implemented.

Giac Verification phase not currently implemented.

Mupad Verification phase not currently implemented.

1.12 Timing

The command `AbsoluteTiming[]` was used in Mathematica to obtain the elapsed time for each integrate call. In Maple, the command `Usage` was used as in the following example

```
cpu_time := Usage(assign ('result_of_int',int(expr,x)),output='realtime'
```

For all other CAS systems, the elapsed time to complete each integral was found by taking the difference between the time after the call completed from the time before the call was made. This was done using Python's `time.time()` call.

All elapsed times shown are in seconds. A time limit of 3 CPU minutes was used for each integral. If the integrate command did not complete within this time limit, the integral was aborted and considered to have failed and assigned an F grade. The time used by failed integrals due to time out was not counted in the final statistics.

1.13 Verification

A verification phase was applied on the result of integration for `Rubi` and `Mathematica`.

Future version of this report will implement verification for the other CAS systems. For the integrals whose result was not run through a verification phase, it is assumed that the antiderivative was correct.

Verification phase also had 3 minutes time out. An integral whose result was not verified could still be correct, but further investigation is needed on those integrals. These integrals were marked in the summary table below and also in each integral separate section so they are easy to identify and locate.

1.14 Important notes about some of the results

1.14.1 Important note about Maxima results

Since tests were run in a batch mode, and using an automated script, then any integral where Maxima needed an interactive response from the user to answer a question during the evaluation of the integral will fail.

The exception raised is `ValueError`. Therefore Maxima results is lower than what would result if Maxima was run directly and each question was answered correctly.

The percentage of such failures were not counted for each test file, but for an example, for the `Timofeev` test file, there were about 14 such integrals out of total 705, or about 2 percent. This percentage can be higher or lower depending on the specific input test file.

Such integrals can be identified by looking at the output of the integration in each section for Maxima. The exception message will indicate the cause of error.

Maxima integrate was run using SageMath with the following settings set by default

```
'besselexpand : true'
'display2d : false'
'domain : complex'
'keepfloat : true'
'load(to_poly_solve)'
'load(simplify_sum)'
'load(abs_integrate)' 'load(diag)'
```

SageMath automatic loading of Maxima `abs_integrate` was found to cause some problems. So the following code was added to disable this effect.

```
from sage.interfaces.maxima_lib import maxima_lib
maxima_lib.set('extra_definite_integration_methods', '[]')
maxima_lib.set('extra_integration_methods', '[]')
```

See `https://ask.sagemath.org/question/43088/integrate-results-that-are-different-from-using-maxima/` for reference.

1.14.2 Important note about FriCAS result

There were few integrals which failed due to SageMath interface and not because FriCAS system could not do the integration.

These will fail With error `Exception raised: NotImplementedError`.

The number of such cases seems to be very small. About 1 or 2 percent of all integrals. These can be identified by looking at the exception message given in the result.

1.14.3 Important note about finding leaf size of antiderivative

For Mathematica, Rubi, and Maple, the builtin system function `LeafSize` was used to find the leaf size of each antiderivative.

The other CAS systems (SageMath and Sympy) do not have special builtin function for this purpose at this time. Therefore the leaf size for Fricas and Sympy antiderivative was determined using the following function, thanks to user `slelievre` at `https://ask.sagemath.org/question/57123/could-we-have-a-leaf_count-function-in-base-sagemath/`

```
def tree_size(expr):
    r"""
    Return the tree size of this expression.
    """
    if expr not in SR:
        # deal with lists, tuples, vectors
        return 1 + sum(tree_size(a) for a in expr)
    expr = SR(expr)
    x, aa = expr.operator(), expr.operands()
    if x is None:
        return 1
    else:
        return 1 + sum(tree_size(a) for a in aa)
```

For Sympy, which was called directly from Python, the following code was used to obtain the leafsize of its result

```
try:
  # 1.7 is a fudge factor since it is low side from actual leaf count
  leafCount = round(1.7*count_ops(anti))

  except Exception as ee:
        leafCount =1
```

1.14.4 Important note about Mupad results

Matlab's symbolic toolbox does not have a leaf count function to measure the size of the antiderivative. Maple was used to determine the leaf size of Mupad output by post processing Mupad result.

Currently no grading of the antiderivative for Mupad is implemented. If it can integrate the problem, it was assigned a B grade automatically as a placeholder. In the future, when grading function is implemented for Mupad, the tests will be rerun again.

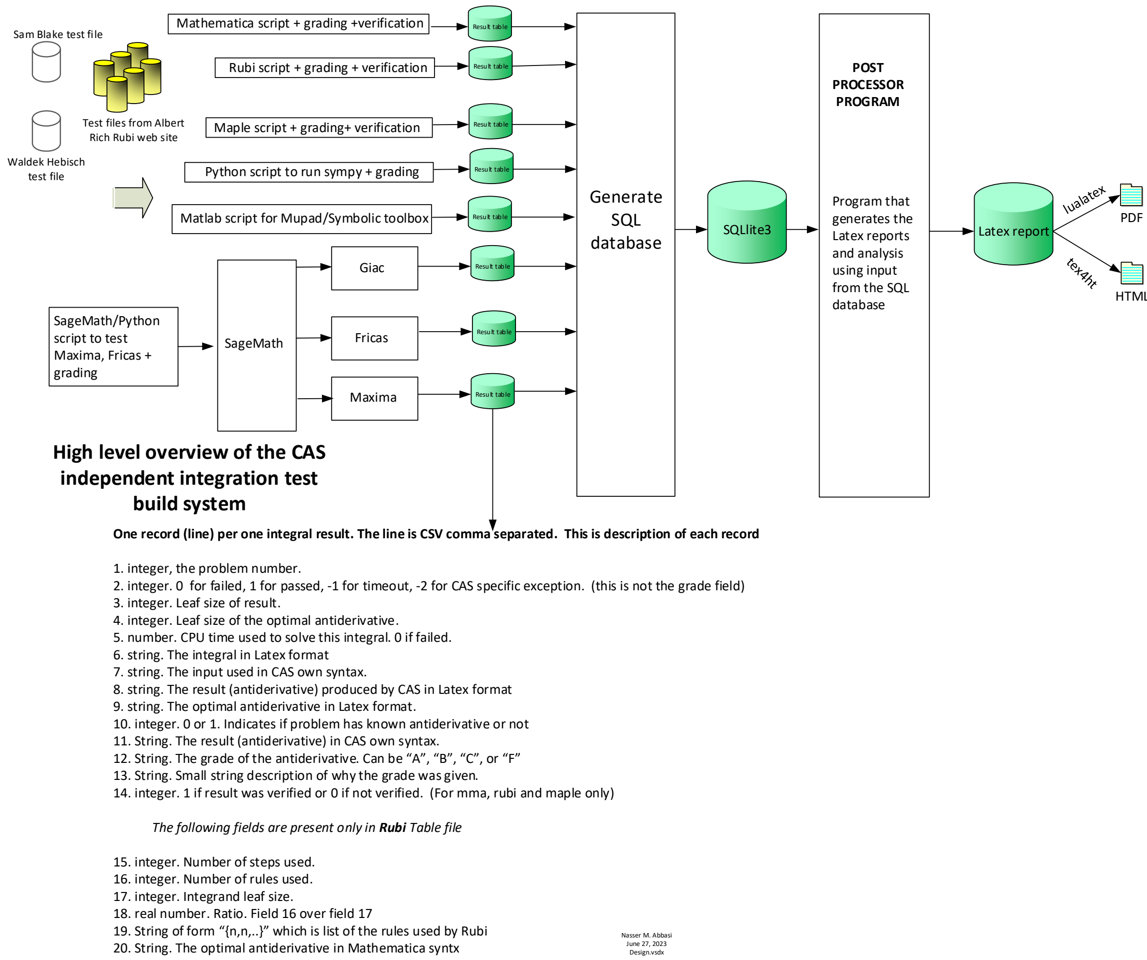
The following is an example of using Matlab's symbolic toolbox (Mupad) to solve an integral

```
integrand = evalin(symengine,'cos(x)*sin(x)')
the_variable = evalin(symengine,'x')
anti = int(integrand,the_variable)
```

Which gives `sin(x)^2/2`

1.15 Design of the test system

The following diagram gives a high level view of the current test build system.



CHAPTER 2

DETAILED SUMMARY TABLES OF RESULTS

2.1 List of integrals sorted by grade for each CAS

2.1.1 Rubi

A grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51 }

B grade { }

C grade { }

F normal fail { }

F(-1) timedout fail { }

F(-2) exception fail { }

2.1.2 Mma

A grade { 1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 15, 16, 17, 18, 19, 20, 21, 22, 27, 28, 39, 48, 49, 50 }

B grade { 8, 9, 32, 33, 37, 38, 45 }

C grade { 13, 14, 23, 24, 25, 26, 29, 30, 31, 34, 35, 36, 40, 41, 42, 46 }

F normal fail { 43, 44, 47, 51 }

F(-1) timedout fail { }

F(-2) exception fail { }

2.1.3 Maple

A grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21, 23, 24, 29, 30, 39, 40, 41 }

B grade { 15, 22, 25, 26, 27, 28, 32, 33, 34, 35, 36, 37, 38, 43, 44, 45, 47, 49, 50 }

C grade { 31, 42, 46, 48 }

F normal fail { 51 }

F(-1) timedout fail { }

F(-2) exception fail { }

2.1.4 Fricas

A grade { 1, 2, 3, 4, 5, 6, 12, 15, 16, 17, 18, 19, 22, 27 }

B grade { 7, 8, 9, 10, 11, 13, 14, 23, 24, 25, 26, 28, 35, 36, 37, 38, 39 }

C grade { }

F normal fail { 20, 21, 31, 32, 33, 34, 45, 51 }

F(-1) timedout fail { 29, 30, 40, 41, 42, 43, 44, 46, 47, 48, 49, 50 }

F(-2) exception fail { }

2.1.5 Maxima

A grade { 1, 2, 3, 4, 5, 6, 7, 10, 11, 20, 21 }

B grade { 8, 9, 16, 17 }

C grade { 22 }

F normal fail { 12, 13, 14, 15, 18, 19, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51 }

F(-1) timedout fail { }

F(-2) exception fail { 39 }

2.1.6 Giac

A grade { 1, 2, 3, 4, 5, 7, 9, 10, 12, 13, 14, 19, 20, 21, 23, 24, 39 }

B grade { 6, 8, 11 }

C grade { }

F normal fail { 16, 26, 29, 30, 31, 32, 33, 34, 36, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51 }

F(-1) timedout fail { 15, 17, 18, 25, 27, 35, 37, 38 }

F(-2) exception fail { 22, 28 }

2.1.7 Mupad

A grade { }

B grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 16, 39 }

C grade { }

F normal fail { }

F(-1) timedout fail { 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51 }

F(-2) exception fail { }

2.1.8 Sympy

A grade { }

B grade { 1, 2, 3, 4 }

C grade { }

F normal fail { 5, 6, 7, 8, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50 }

F(-1) timedout fail { 9, 10, 11, 39, 51 }

F(-2) exception fail { }

2.2 Detailed conclusion table per each integral for all CAS systems

Detailed conclusion table per each integral is given by the table below. The elapsed time is in seconds. For failed result it is given as **F(-1)** if the failure was due to timeout. It is given as **F(-2)** if the failure was due to an exception being raised, which could indicate a bug in the system. If the failure was due to integral not being evaluated within the time limit, then it is given as **F**.

In this table, the column `N.S.` means **normalized size** and is defined as $\frac{\text{antiderivative leaf size}}{\text{optimal antiderivative leaf size}}$. To make the table fit the page, the name `Mathematica` was abbreviated to `MMA`.

Problem 1	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	B	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	121	121	77	79	147	91	415	113	256
N.S.	1	1.00	0.64	0.65	1.21	0.75	3.43	0.93	2.12
time (sec)	N/A	0.376	0.295	1.326	0.193	0.268	0.374	0.439	14.083

Problem 2	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	B	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	96	96	57	57	123	77	301	77	212
N.S.	1	1.00	0.59	0.59	1.28	0.80	3.14	0.80	2.21
time (sec)	N/A	0.340	0.127	1.057	0.201	0.274	0.260	0.381	14.235

Problem 3	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	B	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	77	77	47	46	100	63	245	59	250
N.S.	1	1.00	0.61	0.60	1.30	0.82	3.18	0.77	3.25
time (sec)	N/A	0.290	0.111	0.789	0.199	0.267	0.179	0.367	14.057

Problem 4	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	B	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	52	47	43	44	77	46	133	59	125
N.S.	1	0.90	0.83	0.85	1.48	0.88	2.56	1.13	2.40
time (sec)	N/A	0.307	0.584	0.604	0.198	0.263	0.127	0.595	14.523

Problem 5	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	63	63	61	45	73	75	0	105	88
N.S.	1	1.00	0.97	0.71	1.16	1.19	0.00	1.67	1.40
time (sec)	N/A	0.289	0.136	0.463	0.201	0.271	0.000	0.331	12.674

Problem 6	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	F	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	53	44	97	61	69	99	0	129	110
N.S.	1	0.83	1.83	1.15	1.30	1.87	0.00	2.43	2.08
time (sec)	N/A	0.357	0.080	0.529	0.198	0.277	0.000	0.322	12.173

Problem 7	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	B	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	64	64	95	73	105	138	0	116	163
N.S.	1	1.00	1.48	1.14	1.64	2.16	0.00	1.81	2.55
time (sec)	N/A	0.312	0.627	0.776	0.200	0.283	0.000	0.318	12.172

Problem 8	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	B	A	B	B	F	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	61	64	172	95	120	137	0	139	132
N.S.	1	1.05	2.82	1.56	1.97	2.25	0.00	2.28	2.16
time (sec)	N/A	0.544	0.197	0.916	0.197	0.277	0.000	0.312	11.687

Problem 9	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	B	A	B	B	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	86	86	179	109	165	166	0	140	133
N.S.	1	1.00	2.08	1.27	1.92	1.93	0.00	1.63	1.55
time (sec)	N/A	0.345	0.225	1.070	0.201	0.260	0.000	0.403	11.530

Problem 10	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	B	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	105	105	204	123	187	201	0	174	244
N.S.	1	1.00	1.94	1.17	1.78	1.91	0.00	1.66	2.32
time (sec)	N/A	0.369	0.109	1.242	0.201	0.268	0.000	0.331	12.003

Problem 11	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	B	F(-1)	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	130	130	204	125	232	240	0	242	340
N.S.	1	1.00	1.57	0.96	1.78	1.85	0.00	1.86	2.62
time (sec)	N/A	0.401	0.125	1.327	0.204	0.267	0.000	0.383	12.226

Problem 12	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	F	A	F	A	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	128	185	69	78	0	155	0	99	0
N.S.	1	1.45	0.54	0.61	0.00	1.21	0.00	0.77	0.00
time (sec)	N/A	0.905	1.746	1.108	0.000	0.255	0.000	0.354	0.000

Problem 13	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	F	B	F	A	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	69	69	46	78	0	202	0	110	0
N.S.	1	1.00	0.67	1.13	0.00	2.93	0.00	1.59	0.00
time (sec)	N/A	0.720	1.700	0.513	0.000	0.278	0.000	0.344	0.000

Problem 14	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	F	B	F	A	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	120	127	74	124	0	328	0	141	0
N.S.	1	1.06	0.62	1.03	0.00	2.73	0.00	1.18	0.00
time (sec)	N/A	1.087	0.323	0.391	0.000	0.288	0.000	0.350	0.000

Problem 15	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	B	F	A	F	F(-1)	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD
size	103	103	91	754	0	442	0	0	0
N.S.	1	1.00	0.88	7.32	0.00	4.29	0.00	0.00	0.00
time (sec)	N/A	0.847	2.595	3.211	0.000	0.410	0.000	0.000	0.000

Problem 16	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	B	A	F	F	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	43	43	40	37	309	41	0	0	52
N.S.	1	1.00	0.93	0.86	7.19	0.95	0.00	0.00	1.21
time (sec)	N/A	0.370	0.548	2.859	0.312	0.273	0.000	0.000	13.129

Problem 17	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	B	A	F	F(-1)	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	114	114	133	158	270	385	0	0	0
N.S.	1	1.00	1.17	1.39	2.37	3.38	0.00	0.00	0.00
time (sec)	N/A	0.864	0.219	3.342	0.320	0.360	0.000	0.000	0.000

Problem 18	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	F	A	F	F(-1)	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	118	118	132	130	0	391	0	0	0
N.S.	1	1.00	1.12	1.10	0.00	3.31	0.00	0.00	0.00
time (sec)	N/A	0.874	0.182	3.396	0.000	0.372	0.000	0.000	0.000

Problem 19	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	F	A	F	A	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	46	48	62	68	0	202	0	57	0
N.S.	1	1.04	1.35	1.48	0.00	4.39	0.00	1.24	0.00
time (sec)	N/A	0.408	0.072	1.565	0.000	0.380	0.000	0.336	0.000

Problem 20	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	F	F	A	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	102	105	64	100	59	0	0	1	0
N.S.	1	1.03	0.63	0.98	0.58	0.00	0.00	0.01	0.00
time (sec)	N/A	1.196	1.255	1.226	0.297	0.000	0.000	0.338	0.000

Problem 21	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	F	F	A	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	100	104	61	100	59	0	0	112	0
N.S.	1	1.04	0.61	1.00	0.59	0.00	0.00	1.12	0.00
time (sec)	N/A	1.179	0.774	1.243	0.296	0.000	0.000	0.338	0.000

Problem 22	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	B	C	A	F	F(-2)	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	46	46	63	91	190	193	0	0	0
N.S.	1	1.00	1.37	1.98	4.13	4.20	0.00	0.00	0.00
time (sec)	N/A	0.411	0.105	1.793	0.402	0.328	0.000	0.000	0.000

Problem 23	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	F	B	F	A	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	105	105	746	120	0	781	0	143	0
N.S.	1	1.00	7.10	1.14	0.00	7.44	0.00	1.36	0.00
time (sec)	N/A	0.536	6.013	0.582	0.000	0.581	0.000	0.343	0.000

Problem 24	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	F	B	F	A	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	165	176	654	206	0	1044	0	189	0
N.S.	1	1.07	3.96	1.25	0.00	6.33	0.00	1.15	0.00
time (sec)	N/A	0.995	3.366	0.590	0.000	1.513	0.000	0.396	0.000

Problem 25	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	B	F	B	F	F(-1)	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD
size	149	149	662	935	0	3273	0	0	0
N.S.	1	1.00	4.44	6.28	0.00	21.97	0.00	0.00	0.00
time (sec)	N/A	0.850	3.395	3.638	0.000	1.475	0.000	0.000	0.000

Problem 26	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	B	F	B	F	F	F(-1)
verified	N/A	Yes	No	No	TBD	TBD	TBD	TBD	TBD
size	83	83	436	505	0	1303	0	0	0
N.S.	1	1.00	5.25	6.08	0.00	15.70	0.00	0.00	0.00
time (sec)	N/A	0.389	2.711	3.166	0.000	0.887	0.000	0.000	0.000

Problem 27	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	B	F	A	F	F(-1)	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD
size	166	166	280	589	0	3048	0	0	0
N.S.	1	1.00	1.69	3.55	0.00	18.36	0.00	0.00	0.00
time (sec)	N/A	0.897	5.903	3.142	0.000	1.330	0.000	0.000	0.000

Problem 28	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	B	F	B	F	F(-2)	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD
size	168	168	289	580	0	3175	0	0	0
N.S.	1	1.00	1.72	3.45	0.00	18.90	0.00	0.00	0.00
time (sec)	N/A	0.886	8.216	3.490	0.000	1.576	0.000	0.000	0.000

Problem 29	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	F	F(-1)	F	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	238	273	452	593	0	0	0	0	0
N.S.	1	1.15	1.90	2.49	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	1.771	19.055	1.324	0.000	0.000	0.000	0.000	0.000

Problem 30	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	F	F(-1)	F	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	246	263	462	587	0	0	0	0	0
N.S.	1	1.07	1.88	2.39	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	1.711	5.986	1.252	0.000	0.000	0.000	0.000	0.000

Problem 31	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	C	F	F	F	F	F(-1)
verified	N/A	Yes	No	No	TBD	TBD	TBD	TBD	TBD
size	267	267	13199	9956	0	0	0	0	0
N.S.	1	1.00	49.43	37.29	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.961	32.783	2.604	0.000	0.000	0.000	0.000	0.000

Problem 32	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	B	B	F	F	F	F	F(-1)
verified	N/A	Yes	No	No	TBD	TBD	TBD	TBD	TBD
size	116	116	3415	3007	0	0	0	0	0
N.S.	1	1.00	29.44	25.92	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.401	28.982	2.158	0.000	0.000	0.000	0.000	0.000

Problem 33	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	B	B	F	F	F	F	F(-1)
verified	N/A	Yes	No	No	TBD	TBD	TBD	TBD	TBD
size	252	252	4464	3057	0	0	0	0	0
N.S.	1	1.00	17.71	12.13	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.948	28.851	2.485	0.000	0.000	0.000	0.000	0.000

Problem 34	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	B	F	F	F	F	F(-1)
verified	N/A	Yes	No	No	TBD	TBD	TBD	TBD	TBD
size	256	256	1667	3938	0	0	0	0	0
N.S.	1	1.00	6.51	15.38	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.969	19.746	2.612	0.000	0.000	0.000	0.000	0.000

Problem 35	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	B	F	B	F	F(-1)	F(-1)
verified	N/A	Yes	No	No	TBD	TBD	TBD	TBD	TBD
size	123	123	567	5066	0	3539	0	0	0
N.S.	1	1.00	4.61	41.19	0.00	28.77	0.00	0.00	0.00
time (sec)	N/A	0.878	2.899	0.841	0.000	1.018	0.000	0.000	0.000

Problem 36	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	B	F	B	F	F	F(-1)
verified	N/A	Yes	No	Yes	TBD	TBD	TBD	TBD	TBD
size	61	61	367	203	0	1044	0	0	0
N.S.	1	1.00	6.02	3.33	0.00	17.11	0.00	0.00	0.00
time (sec)	N/A	0.368	1.836	2.044	0.000	0.456	0.000	0.000	0.000

Problem 37	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	B	B	F	B	F	F(-1)	F(-1)
verified	N/A	Yes	No	No	TBD	TBD	TBD	TBD	TBD
size	140	140	665	326	0	2791	0	0	0
N.S.	1	1.00	4.75	2.33	0.00	19.94	0.00	0.00	0.00
time (sec)	N/A	0.921	10.589	1.805	0.000	0.611	0.000	0.000	0.000

Problem 38	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	B	B	F	B	F	F(-1)	F(-1)
verified	N/A	Yes	No	No	TBD	TBD	TBD	TBD	TBD
size	140	140	885	347	0	3005	0	0	0
N.S.	1	1.00	6.32	2.48	0.00	21.46	0.00	0.00	0.00
time (sec)	N/A	0.886	14.913	1.648	0.000	0.711	0.000	0.000	0.000

Problem 39	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	F(-2)	B	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	181	225	178	244	0	2837	0	292	23933
N.S.	1	1.24	0.98	1.35	0.00	15.67	0.00	1.61	132.23
time (sec)	N/A	0.864	1.321	2.033	0.000	65.065	0.000	0.441	27.413

Problem 40	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	F	F(-1)	F	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	154	154	179	190	0	0	0	0	0
N.S.	1	1.00	1.16	1.23	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.830	30.020	1.286	0.000	0.000	0.000	0.000	0.000

Problem 41	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	F	F(-1)	F	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	146	146	203	254	0	0	0	0	0
N.S.	1	1.00	1.39	1.74	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.850	29.589	1.148	0.000	0.000	0.000	0.000	0.000

Problem 42	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	C	F	F(-1)	F	F	F(-1)
verified	N/A	Yes	No	No	TBD	TBD	TBD	TBD	TBD
size	254	254	23019	5478	0	0	0	0	0
N.S.	1	1.00	90.63	21.57	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.977	31.903	2.660	0.000	0.000	0.000	0.000	0.000

Problem 43	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	F	B	F	F(-1)	F	F	F(-1)
verified	N/A	Yes	N/A	No	TBD	TBD	TBD	TBD	TBD
size	250	250	0	2884	0	0	0	0	0
N.S.	1	1.00	0.00	11.54	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.969	0.000	2.502	0.000	0.000	0.000	0.000	0.000

Problem 44	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	F	B	F	F(-1)	F	F	F(-1)
verified	N/A	Yes	N/A	No	TBD	TBD	TBD	TBD	TBD
size	114	114	0	2583	0	0	0	0	0
N.S.	1	1.00	0.00	22.66	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.410	0.000	2.800	0.000	0.000	0.000	0.000	0.000

Problem 45	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	B	B	F	F	F	F	F(-1)
verified	N/A	Yes	No	No	TBD	TBD	TBD	TBD	TBD
size	246	246	5612	3343	0	0	0	0	0
N.S.	1	1.00	22.81	13.59	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.988	31.093	2.860	0.000	0.000	0.000	0.000	0.000

Problem 46	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	C	F	F(-1)	F	F	F(-1)
verified	N/A	Yes	No	No	TBD	TBD	TBD	TBD	TBD
size	254	254	23019	5489	0	0	0	0	0
N.S.	1	1.00	90.63	21.61	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.978	32.295	2.691	0.000	0.000	0.000	0.000	0.000

Problem 47	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	F	B	F	F(-1)	F	F	F(-1)
verified	N/A	Yes	N/A	No	TBD	TBD	TBD	TBD	TBD
size	114	114	0	2590	0	0	0	0	0
N.S.	1	1.00	0.00	22.72	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.427	0.000	2.925	0.000	0.000	0.000	0.000	0.000

Problem 48	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	F	F(-1)	F	F	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD
size	391	391	274	242134	0	0	0	0	0
N.S.	1	1.00	0.70	619.27	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	1.107	0.192	7.630	0.000	0.000	0.000	0.000	0.000

Problem 49	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	B	F	F(-1)	F	F	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD
size	198	198	197	28726	0	0	0	0	0
N.S.	1	1.00	0.99	145.08	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.454	0.112	7.148	0.000	0.000	0.000	0.000	0.000

Problem 50	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	B	F	F(-1)	F	F	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD
size	398	398	374	24290	0	0	0	0	0
N.S.	1	1.00	0.94	61.03	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	1.100	1.618	7.125	0.000	0.000	0.000	0.000	0.000

Problem 51	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	F	F	F	F	F(-1)	F	F(-1)
verified	N/A	Yes	N/A	N/A	TBD	TBD	TBD	TBD	TBD
size	157	157	0	0	0	0	0	0	0
N.S.	1	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.464	0.000	0.000	0.000	0.000	0.000	0.000	0.000

2.3 Detailed conclusion table specific for Rubi results

The following table is specific to Rubi only. It gives additional statistics for each integral. the column **steps** is the number of steps used by Rubi to obtain the antiderivative. The **rules** column is the number of unique rules used. The **integrand size** column is the leaf size of the integrand. Finally the ratio $\frac{\text{number of rules}}{\text{integrand size}}$ is also given. The larger this ratio is, the harder the integral is to solve. In this test file, problem number [29] had the largest ratio of [.515152000000000054]

Table 2.1: Rubi specific breakdown of results for each integral

#	grade	number of steps used	number of unique rules	normalized antiderivative leaf size	integrand leaf size	$\frac{\text{number of rules}}{\text{integrand leaf size}}$
1	A	3	3	1.00	32	0.094
2	A	3	3	1.00	32	0.094
3	A	3	3	1.00	30	0.100
4	A	7	7	0.90	24	0.292
5	A	3	3	1.00	30	0.100
6	A	5	5	0.83	32	0.156
7	A	3	3	1.00	32	0.094
8	A	11	10	1.05	32	0.312
9	A	3	3	1.00	32	0.094
10	A	3	3	1.00	32	0.094
11	A	3	3	1.00	32	0.094
12	A	12	12	1.45	34	0.353
13	A	9	8	1.00	34	0.235
14	A	13	12	1.06	34	0.353
15	A	8	7	1.00	40	0.175
16	A	4	3	1.00	40	0.075
17	A	8	7	1.00	40	0.175
18	A	8	7	1.00	40	0.175
19	A	5	5	1.04	36	0.139
20	A	12	11	1.03	36	0.306
21	A	12	11	1.04	36	0.306
22	A	6	5	1.00	36	0.139

Continued on next page

Table 2.1 – continued from previous page

#	grade	number of steps used	number of unique rules	normalized antiderivative leaf size	integrand leaf size	$\frac{\text{number of rules}}{\text{integrand leaf size}}$
23	A	7	6	1.00	33	0.182
24	A	12	11	1.07	33	0.333
25	A	8	7	1.00	39	0.179
26	A	4	3	1.00	39	0.077
27	A	8	7	1.00	39	0.179
28	A	8	7	1.00	39	0.179
29	A	17	17	1.15	33	0.515
30	A	17	17	1.07	33	0.515
31	A	5	5	1.00	39	0.128
32	A	2	2	1.00	39	0.051
33	A	5	5	1.00	39	0.128
34	A	5	5	1.00	39	0.128
35	A	8	7	1.00	35	0.200
36	A	4	3	1.00	35	0.086
37	A	8	7	1.00	35	0.200
38	A	8	7	1.00	35	0.200
39	A	10	9	1.24	33	0.273
40	A	6	6	1.00	33	0.182
41	A	6	6	1.00	33	0.182
42	A	5	5	1.00	39	0.128
43	A	5	5	1.00	39	0.128
44	A	2	2	1.00	39	0.051
45	A	5	5	1.00	39	0.128
46	A	5	5	1.00	39	0.128
47	A	2	2	1.00	39	0.051
48	A	5	5	1.00	35	0.143
49	A	2	2	1.00	35	0.057
50	A	5	5	1.00	35	0.143
51	A	6	5	1.00	38	0.132

2.3. Detailed conclusion table specific for Rubi results

CHAPTER 3

Listing of integrals

3.1 $\int \sin^3(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$

3.1.1 Optimal result

Integrand size = 32, antiderivative size = 121

$$\begin{aligned}\int \sin^3(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx \\ =\frac{1}{16}a^2cx-\frac{a^2c\cos^3(e+fx)}{3f}+\frac{a^2c\cos^5(e+fx)}{5f}-\frac{a^2c\cos(e+fx)\sin(e+fx)}{16f} \\ -\frac{a^2c\cos(e+fx)\sin^3(e+fx)}{24f}+\frac{a^2c\cos(e+fx)\sin^5(e+fx)}{6f}\end{aligned}$$

output
```
1/16*a^2*c*x-1/3*a^2*c*cos(f*x+e)^3/f+1/5*a^2*c*cos(f*x+e)^5/f-1/16*a^2*c*
cos(f*x+e)*sin(f*x+e)/f-1/24*a^2*c*cos(f*x+e)*sin(f*x+e)^3/f+1/6*a^2*c*cos
(f*x+e)*sin(f*x+e)^5/f
```

3.1.2 Mathematica [A] (verified)

Time = 0.29 (sec) , antiderivative size = 77, normalized size of antiderivative = 0.64

$$\begin{aligned}\int \sin^3(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx \\ =\frac{a^2c(60e+60fx-120\cos(e+fx)-20\cos(3(e+fx))+12\cos(5(e+fx))-15\sin(2(e+fx))-15\sin(}{960f}\end{aligned}$$

input
```
Integrate[Sin[e + f*x]^3*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output
```
(a^2*c*(60*e + 60*f*x - 120*Cos[e + f*x] - 20*Cos[3*(e + f*x)] + 12*Cos[5*
(e + f*x)] - 15*Sin[2*(e + f*x)] - 15*Sin[4*(e + f*x)] + 5*Sin[6*(e + f*x)
]))/(960*f)
```

3.1.3 Rubi [A] (verified)

Time = 0.38 (sec) , antiderivative size = 121, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.094$, Rules used = {3042, 3445, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \sin^3(e+fx)(a\sin(e+fx)+a)^2(c-c\sin(e+fx))\,dx$$

$$\Big\downarrow \text{ 3042}$$

$$\int \sin(e+fx)^3(a\sin(e+fx)+a)^2(c-c\sin(e+fx))dx$$

$$\Big\downarrow \text{ 3445}$$

$$\int \left(-a^2c\sin^6(e+fx)-a^2c\sin^5(e+fx)+a^2c\sin^4(e+fx)+a^2c\sin^3(e+fx)\right)dx$$

$$\Big\downarrow \text{ 2009}$$

$$\frac{a^2c\cos^5(e+fx)}{5f}-\frac{a^2c\cos^3(e+fx)}{3f}+\frac{a^2c\sin^5(e+fx)\cos(e+fx)}{6f}-\frac{a^2c\sin^3(e+fx)\cos(e+fx)}{24f}-\frac{a^2c\sin(e+fx)\cos(e+fx)}{16f}+\frac{1}{16}a^2cx$$

input
```
Int[Sin[e + f*x]^3*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output
```
(a^2*c*x)/16 - (a^2*c*Cos[e + f*x]^3)/(3*f) + (a^2*c*Cos[e + f*x]^5)/(5*f)
 - (a^2*c*Cos[e + f*x]*Sin[e + f*x])/(16*f) - (a^2*c*Cos[e + f*x]*Sin[e +
f*x]^3)/(24*f) + (a^2*c*Cos[e + f*x]*Sin[e + f*x]^5)/(6*f)
```

3.1.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3445
```
Int[sin[(e_.) + (f_.)*(x_)]^(n_.)*((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m
_.)*((A_.) + (B_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] :> Int[ExpandTrig[si
n[e + f*x]^n*(a + b*sin[e + f*x])^m*(A + B*sin[e + f*x]), x], x] /; FreeQ[{
a, b, e, f, A, B}, x] && EqQ[A*b + a*B, 0] && EqQ[a^2 - b^2, 0] && IntegerQ
[m] && IntegerQ[n]
```

3.1.4 Maple [A] (verified)

Time = 1.33 (sec) , antiderivative size = 79, normalized size of antiderivative = 0.65

method	result
parallelrisch	$-\frac{a^2c(-60fx+120\cos(fx+e)-5\sin(6fx+6e)-12\cos(5fx+5e)+15\sin(4fx+4e)+20\cos(3fx+3e)+15\sin(2fx+2e)+1}{960f}$
risch	$\frac{a^2cx}{16}-\frac{a^2c\cos(fx+e)}{8f}+\frac{a^2c\sin(6fx+6e)}{192f}+\frac{a^2c\cos(5fx+5e)}{80f}-\frac{a^2c\sin(4fx+4e)}{64f}-\frac{a^2c\cos(3fx+3e)}{48f}-\frac{a^2cs}{}$
derivativedivides	$\frac{-a^2c\left(-\frac{\left(\sin^5(fx+e)+\frac{5(\sin^3(fx+e))}{4}+\frac{15\sin(fx+e)}{8}\right)\cos(fx+e)}{6}+\frac{5fx}{16}+\frac{5e}{16}\right)+\frac{a^2c\left(\frac{8}{3}+\sin^4(fx+e)+\frac{4(\sin^2(fx+e))}{3}\right)\cos(fx}{5}}{f}$
default	$\frac{-a^2c\left(-\frac{\left(\sin^5(fx+e)+\frac{5(\sin^3(fx+e))}{4}+\frac{15\sin(fx+e)}{8}\right)\cos(fx+e)}{6}+\frac{5fx}{16}+\frac{5e}{16}\right)+\frac{a^2c\left(\frac{8}{3}+\sin^4(fx+e)+\frac{4(\sin^2(fx+e))}{3}\right)\cos(fx}{5}}{f}$
parts	$-\frac{a^2c(2+\sin^2(fx+e))\cos(fx+e)}{3f}+\frac{a^2c\left(-\frac{\left(\sin^3(fx+e)+\frac{3\sin(fx+e)}{2}\right)\cos(fx+e)}{4}+\frac{3fx}{8}+\frac{3e}{8}\right)}{f}+\frac{a^2c\left(\frac{8}{3}+\sin^4(fx+e)\right.}{}$
norman	$\frac{-\frac{4a^2c}{15f}+\frac{a^2cx}{16}-\frac{8a^2c\left(\tan^6\left(\frac{fx}{2}+\frac{e}{2}\right)\right)}{3f}-\frac{8a^2c\left(\tan^2\left(\frac{fx}{2}+\frac{e}{2}\right)\right)}{5f}-\frac{4a^2c\left(\tan^8\left(\frac{fx}{2}+\frac{e}{2}\right)\right)}{f}-\frac{a^2c\tan\left(\frac{fx}{2}+\frac{e}{2}\right)}{8f}-\frac{17a^2c\left(\tan^3\left(\frac{fx}{2}+\frac{e}{2}\right)\right)}{24f}}{}$

input
```
int(sin(f*x+e)^3*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x,method=_RETURNVERBO
SE)
```

output
```
-1/960*a^2*c*(-60*f*x+120*cos(f*x+e)-5*sin(6*f*x+6*e)-12*cos(5*f*x+5*e)+15
*sin(4*f*x+4*e)+20*cos(3*f*x+3*e)+15*sin(2*f*x+2*e)+128)/f
```

3.1.5 Fricas [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 91, normalized size of antiderivative = 0.75

$$\int \sin^3(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$= \frac{48\,a^2c\cos\left(fx+e\right)^5-80\,a^2c\cos\left(fx+e\right)^3+15\,a^2cfx+5\left(8\,a^2c\cos\left(fx+e\right)^5-14\,a^2c\cos\left(fx+e\right)^3+3\,a\right.}{240\,f}$$

input
```
integrate(sin(f*x+e)^3*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="f
ricas")
```

output
```
1/240*(48*a^2*c*cos(f*x + e)^5 - 80*a^2*c*cos(f*x + e)^3 + 15*a^2*c*f*x +
5*(8*a^2*c*cos(f*x + e)^5 - 14*a^2*c*cos(f*x + e)^3 + 3*a^2*c*cos(f*x + e)
)*sin(f*x + e))/f
```

3.1.6 Sympy [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 415 vs. $2(110) = 220$.

Time = 0.37 (sec) , antiderivative size = 415, normalized size of antiderivative = 3.43

$$\int \sin^3(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$= \begin{cases} -\frac{5a^2cx\sin^6(e+fx)}{16} - \frac{15a^2cx\sin^4(e+fx)\cos^2(e+fx)}{16} + \frac{3a^2cx\sin^4(e+fx)}{8} - \frac{15a^2cx\sin^2(e+fx)\cos^4(e+fx)}{16} + \frac{3a^2cx\sin^2(e+fx)\cos}{4} \\ x(a\sin(e)+a)^2(-c\sin(e)+c)\sin^3(e) \end{cases}$$

input
```
integrate(sin(f*x+e)**3*(a+a*sin(f*x+e))**2*(c-c*sin(f*x+e)),x)
```

output
```
Piecewise((-5*a**2*c*x*sin(e + f*x)**6/16 - 15*a**2*c*x*sin(e + f*x)**4*co
s(e + f*x)**2/16 + 3*a**2*c*x*sin(e + f*x)**4/8 - 15*a**2*c*x*sin(e + f*x)
**2*cos(e + f*x)**4/16 + 3*a**2*c*x*sin(e + f*x)**2*cos(e + f*x)**2/4 - 5*
a**2*c*x*cos(e + f*x)**6/16 + 3*a**2*c*x*cos(e + f*x)**4/8 + 11*a**2*c*sin
(e + f*x)**5*cos(e + f*x)/(16*f) + a**2*c*sin(e + f*x)**4*cos(e + f*x)/f +
 5*a**2*c*sin(e + f*x)**3*cos(e + f*x)**3/(6*f) - 5*a**2*c*sin(e + f*x)**3
*cos(e + f*x)/(8*f) + 4*a**2*c*sin(e + f*x)**2*cos(e + f*x)**3/(3*f) - a**
2*c*sin(e + f*x)**2*cos(e + f*x)/f + 5*a**2*c*sin(e + f*x)*cos(e + f*x)**5
/(16*f) - 3*a**2*c*sin(e + f*x)*cos(e + f*x)**3/(8*f) + 8*a**2*c*cos(e + f
*x)**5/(15*f) - 2*a**2*c*cos(e + f*x)**3/(3*f), Ne(f, 0)), (x*(a*sin(e) +
a)**2*(-c*sin(e) + c)*sin(e)**3, True))
```

3.1.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 147, normalized size of antiderivative = 1.21

$$\int \sin^3(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$= \frac{64\left(3\cos\left(fx+e\right)^5-10\cos\left(fx+e\right)^3+15\cos\left(fx+e\right)\right)a^2c+320\left(\cos\left(fx+e\right)^3-3\cos\left(fx+e\right)\right)a^2c-}{}$$

input
```
integrate(sin(f*x+e)^3*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="m
axima")
```

output
```
1/960*(64*(3*cos(f*x + e)^5 - 10*cos(f*x + e)^3 + 15*cos(f*x + e))*a^2*c +
 320*(cos(f*x + e)^3 - 3*cos(f*x + e))*a^2*c - 5*(4*sin(2*f*x + 2*e)^3 + 6
0*f*x + 60*e + 9*sin(4*f*x + 4*e) - 48*sin(2*f*x + 2*e))*a^2*c + 30*(12*f*
x + 12*e + sin(4*f*x + 4*e) - 8*sin(2*f*x + 2*e))*a^2*c)/f
```

3.1.8 Giac [A] (verification not implemented)

Time = 0.44 (sec) , antiderivative size = 113, normalized size of antiderivative = 0.93

$$\int \sin^3(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$= \frac{1}{16}a^2cx + \frac{a^2c\cos\left(5fx+5e\right)}{80f} - \frac{a^2c\cos\left(3fx+3e\right)}{48f} - \frac{a^2c\cos\left(fx+e\right)}{8f} + \frac{a^2c\sin\left(6fx+6e\right)}{192f} - \frac{a^2c\sin\left(4fx+4e\right)}{64f} - \frac{a^2c\sin\left(2fx+2e\right)}{64f}$$

input
```
integrate(sin(f*x+e)^3*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="g
iac")
```

output
```
1/16*a^2*c*x + 1/80*a^2*c*cos(5*f*x + 5*e)/f - 1/48*a^2*c*cos(3*f*x + 3*e)
/f - 1/8*a^2*c*cos(f*x + e)/f + 1/192*a^2*c*sin(6*f*x + 6*e)/f - 1/64*a^2*
c*sin(4*f*x + 4*e)/f - 1/64*a^2*c*sin(2*f*x + 2*e)/f
```

3.1.9 Mupad [B] (verification not implemented)

Time = 14.08 (sec) , antiderivative size = 256, normalized size of antiderivative = 2.12

$$\int \sin^3(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$= \frac{a^2\,c\,\left(15\,e-30\,\tan\left(\frac{e}{2}+\frac{f\,x}{2}\right)-384\,\tan\left(\frac{e}{2}+\frac{f\,x}{2}\right)^2-170\,\tan\left(\frac{e}{2}+\frac{f\,x}{2}\right)^3+1140\,\tan\left(\frac{e}{2}+\frac{f\,x}{2}\right)^5-640\,\tan\left(\frac{e}{2}+\right.\right.}{}$$

input
```
int(sin(e + f*x)^3*(a + a*sin(e + f*x))^2*(c - c*sin(e + f*x)),x)
```

output
```
(a^2*c*(15*e - 30*tan(e/2 + (f*x)/2) - 384*tan(e/2 + (f*x)/2)^2 - 170*tan(
e/2 + (f*x)/2)^3 + 1140*tan(e/2 + (f*x)/2)^5 - 640*tan(e/2 + (f*x)/2)^6 -
1140*tan(e/2 + (f*x)/2)^7 - 960*tan(e/2 + (f*x)/2)^8 + 170*tan(e/2 + (f*x)
/2)^9 + 30*tan(e/2 + (f*x)/2)^11 + 15*f*x + 90*tan(e/2 + (f*x)/2)^2*(e + f
*x) + 225*tan(e/2 + (f*x)/2)^4*(e + f*x) + 300*tan(e/2 + (f*x)/2)^6*(e + f
*x) + 225*tan(e/2 + (f*x)/2)^8*(e + f*x) + 90*tan(e/2 + (f*x)/2)^10*(e + f
*x) + 15*tan(e/2 + (f*x)/2)^12*(e + f*x) - 64))/(240*f*(tan(e/2 + (f*x)/2)
^2 + 1)^6)
```

3.2 $\int \sin^2(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$

3.2.1 Optimal result

Integrand size = 32, antiderivative size = 96

$$\int \sin^2(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx = \frac{1}{8}a^2cx-\frac{a^2c\cos^3(e+fx)}{3f}+\frac{a^2c\cos^5(e+fx)}{5f} - \frac{a^2c\cos(e+fx)\sin(e+fx)}{8f}+\frac{a^2c\cos(e+fx)\sin^3(e+fx)}{4f}$$

output
```
1/8*a^2*c*x-1/3*a^2*c*cos(f*x+e)^3/f+1/5*a^2*c*cos(f*x+e)^5/f-1/8*a^2*c*co
s(f*x+e)*sin(f*x+e)/f+1/4*a^2*c*cos(f*x+e)*sin(f*x+e)^3/f
```

3.2.2 Mathematica [A] (verified)

Time = 0.13 (sec) , antiderivative size = 57, normalized size of antiderivative = 0.59

$$\int \sin^2(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx = \frac{a^2c(60e+60fx-60\cos(e+fx)-10\cos(3(e+fx))+6\cos(5(e+fx))-15\sin(4(e+fx)))}{480f}$$

input
```
Integrate[Sin[e + f*x]^2*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output
```
(a^2*c*(60*e + 60*f*x - 60*Cos[e + f*x] - 10*Cos[3*(e + f*x)] + 6*Cos[5*(e
 + f*x)] - 15*Sin[4*(e + f*x)]))/(480*f)
```

3.2.3 Rubi [A] (verified)

Time = 0.34 (sec) , antiderivative size = 96, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.094$, Rules used = {3042, 3445, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \sin^2(e+fx)(a\sin(e+fx)+a)^2(c-c\sin(e+fx))\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int \sin(e+fx)^2(a\sin(e+fx)+a)^2(c-c\sin(e+fx))dx$$

$$\Big\downarrow \text{3445}$$

$$\int \left(-a^2c\sin^5(e+fx)-a^2c\sin^4(e+fx)+a^2c\sin^3(e+fx)+a^2c\sin^2(e+fx)\right)dx$$

$$\Big\downarrow \text{2009}$$

$$\frac{a^2c\cos^5(e+fx)}{5f}-\frac{a^2c\cos^3(e+fx)}{3f}+\frac{a^2c\sin^3(e+fx)\cos(e+fx)}{4f}-\frac{a^2c\sin(e+fx)\cos(e+fx)}{8f}+\frac{1}{8}a^2cx$$

input
```
Int[Sin[e + f*x]^2*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output
```
(a^2*c*x)/8 - (a^2*c*Cos[e + f*x]^3)/(3*f) + (a^2*c*Cos[e + f*x]^5)/(5*f)
- (a^2*c*Cos[e + f*x]*Sin[e + f*x])/(8*f) + (a^2*c*Cos[e + f*x]*Sin[e + f*
x]^3)/(4*f)
```

3.2.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3445
```
Int[sin[(e_.) + (f_.)*(x_)]^(n_.)*((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m
_.)*((A_.) + (B_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] :> Int[ExpandTrig[si
n[e + f*x]^n*(a + b*sin[e + f*x])^m*(A + B*sin[e + f*x]), x], x] /; FreeQ[{
a, b, e, f, A, B}, x] && EqQ[A*b + a*B, 0] && EqQ[a^2 - b^2, 0] && IntegerQ
[m] && IntegerQ[n]
```

3.2.4 Maple [A] (verified)

Time = 1.06 (sec) , antiderivative size = 57, normalized size of antiderivative = 0.59

method	result
parallelrisch	$-\frac{a^2 c(-60fx+60\cos(fx+e)-6\cos(5fx+5e)+15\sin(4fx+4e)+10\cos(3fx+3e)+64)}{480f}$
risch	$\frac{a^2cx}{8}-\frac{a^2c\cos(fx+e)}{8f}+\frac{a^2c\cos(5fx+5e)}{80f}-\frac{a^2c\sin(4fx+4e)}{32f}-\frac{a^2c\cos(3fx+3e)}{48f}$
derivativedivides	$\frac{\frac{a^2c\left(\frac{8}{3}+\sin^4(fx+e)+\frac{4\left(\sin^2(fx+e)\right)}{3}\right)\cos(fx+e)}{5}-a^2c\left(-\frac{\left(\sin^3(fx+e)+\frac{3\sin(fx+e)}{2}\right)\cos(fx+e)}{4}+\frac{3fx}{8}+\frac{3e}{8}\right)-\frac{a^2c\left(2+\sin^2(fx\cdots\right.}{}}{f}$
default	$\frac{\frac{a^2c\left(\frac{8}{3}+\sin^4(fx+e)+\frac{4\left(\sin^2(fx+e)\right)}{3}\right)\cos(fx+e)}{5}-a^2c\left(-\frac{\left(\sin^3(fx+e)+\frac{3\sin(fx+e)}{2}\right)\cos(fx+e)}{4}+\frac{3fx}{8}+\frac{3e}{8}\right)-\frac{a^2c\left(2+\sin^2(fx\cdots\right.}{}}{f}$
parts	$\frac{a^2c\left(-\frac{\cos(fx+e)\sin(fx+e)}{2}+\frac{fx}{2}+\frac{e}{2}\right)}{f}-\frac{a^2c(2+\sin^2(fx+e))\cos(fx+e)}{3f}-\frac{a^2c\left(-\frac{\left(\sin^3(fx+e)+\frac{3\sin(fx+e)}{2}\right)\cos(fx+e\cdots}{4}\right.}{f}$
norman	$\frac{-\frac{4a^2c}{15f}+\frac{a^2cx}{8}-\frac{4a^2c\left(\tan^6\left(\frac{fx}{2}+\frac{e}{2}\right)\right)}{f}+\frac{4a^2c\left(\tan^4\left(\frac{fx}{2}+\frac{e}{2}\right)\right)}{3f}-\frac{4a^2c\left(\tan^2\left(\frac{fx}{2}+\frac{e}{2}\right)\right)}{3f}-\frac{a^2c\tan\left(\frac{fx}{2}+\frac{e}{2}\right)}{4f}+\frac{3a^2c\left(\tan^3\left(\frac{fx}{2}+\frac{e}{2}\right)\right)}{2f}-\cdots}{\cdots}$

input
```
int(sin(f*x+e)^2*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x,method=_RETURNVERBO
SE)
```

output
```
-1/480*a^2*c*(-60*f*x+60*cos(f*x+e)-6*cos(5*f*x+5*e)+15*sin(4*f*x+4*e)+10*
cos(3*f*x+3*e)+64)/f
```

3.2.5 Fricas [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 77, normalized size of antiderivative = 0.80

$$\int \sin^2(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$= \frac{24\,a^2c\cos\left(fx+e\right)^5 - 40\,a^2c\cos\left(fx+e\right)^3 + 15\,a^2cfx - 15\left(2\,a^2c\cos\left(fx+e\right)^3 - a^2c\cos\left(fx+e\right)\right)\sin\left(f\right.}{120\,f}$$

input
```
integrate(sin(f*x+e)^2*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="f
ricas")
```

output
```
1/120*(24*a^2*c*cos(f*x + e)^5 - 40*a^2*c*cos(f*x + e)^3 + 15*a^2*c*f*x -
15*(2*a^2*c*cos(f*x + e)^3 - a^2*c*cos(f*x + e))*sin(f*x + e))/f
```

3.2.6 Sympy [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 301 vs. 2(87) = 174.

Time = 0.26 (sec) , antiderivative size = 301, normalized size of antiderivative = 3.14

$$\int \sin^2(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$= \begin{cases} -\frac{3a^2cx\sin^4(e+fx)}{8} - \frac{3a^2cx\sin^2(e+fx)\cos^2(e+fx)}{4} + \frac{a^2cx\sin^2(e+fx)}{2} - \frac{3a^2cx\cos^4(e+fx)}{8} + \frac{a^2cx\cos^2(e+fx)}{2} + \frac{a^2c\sin^4(e+f}{f} \\ x(a\sin(e)+a)^2(-c\sin(e)+c)\sin^2(e) \end{cases}$$

input
```
integrate(sin(f*x+e)**2*(a+a*sin(f*x+e))**2*(c-c*sin(f*x+e)),x)
```

output
```
Piecewise((-3*a**2*c*x*sin(e + f*x)**4/8 - 3*a**2*c*x*sin(e + f*x)**2*cos(
e + f*x)**2/4 + a**2*c*x*sin(e + f*x)**2/2 - 3*a**2*c*x*cos(e + f*x)**4/8
+ a**2*c*x*cos(e + f*x)**2/2 + a**2*c*sin(e + f*x)**4*cos(e + f*x)/f + 5*a
**2*c*sin(e + f*x)**3*cos(e + f*x)/(8*f) + 4*a**2*c*sin(e + f*x)**2*cos(e
+ f*x)**3/(3*f) - a**2*c*sin(e + f*x)**2*cos(e + f*x)/f + 3*a**2*c*sin(e +
 f*x)*cos(e + f*x)**3/(8*f) - a**2*c*sin(e + f*x)*cos(e + f*x)/(2*f) + 8*a
**2*c*cos(e + f*x)**5/(15*f) - 2*a**2*c*cos(e + f*x)**3/(3*f), Ne(f, 0)),
(x*(a*sin(e) + a)**2*(-c*sin(e) + c)*sin(e)**2, True))
```

3.2.7 Maxima [A] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 123, normalized size of antiderivative = 1.28

$$\int \sin^2(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$= \frac{32\left(3\cos\left(fx+e\right)^5-10\cos\left(fx+e\right)^3+15\cos\left(fx+e\right)\right)a^2c+160\left(\cos\left(fx+e\right)^3-3\cos\left(fx+e\right)\right)a^2c-}{480\,f}$$

input
```
integrate(sin(f*x+e)^2*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="m
axima")
```

output
```
1/480*(32*(3*cos(f*x + e)^5 - 10*cos(f*x + e)^3 + 15*cos(f*x + e))*a^2*c +
 160*(cos(f*x + e)^3 - 3*cos(f*x + e))*a^2*c - 15*(12*f*x + 12*e + sin(4*f
*x + 4*e) - 8*sin(2*f*x + 2*e))*a^2*c + 120*(2*f*x + 2*e - sin(2*f*x + 2*e
))*a^2*c)/f
```

3.2.8 Giac [A] (verification not implemented)

Time = 0.38 (sec) , antiderivative size = 77, normalized size of antiderivative = 0.80

$$\int \sin^2(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$= \frac{1}{8}a^2cx+\frac{a^2c\cos\left(5\,fx+5\,e\right)}{80\,f}-\frac{a^2c\cos\left(3\,fx+3\,e\right)}{48\,f} - \frac{a^2c\cos\left(fx+e\right)}{8\,f}-\frac{a^2c\sin\left(4\,fx+4\,e\right)}{32\,f}$$

input
```
integrate(sin(f*x+e)^2*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="g
iac")
```

output
```
1/8*a^2*c*x + 1/80*a^2*c*cos(5*f*x + 5*e)/f - 1/48*a^2*c*cos(3*f*x + 3*e)/
f - 1/8*a^2*c*cos(f*x + e)/f - 1/32*a^2*c*sin(4*f*x + 4*e)/f
```

3.2.9 Mupad [B] (verification not implemented)

Time = 14.24 (sec) , antiderivative size = 212, normalized size of antiderivative = 2.21

$$\int \sin^2(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$= \frac{a^2\,c\left(15\,e-30\tan\left(\frac{e}{2}+\frac{f\,x}{2}\right)-160\tan\left(\frac{e}{2}+\frac{f\,x}{2}\right)^2+180\tan\left(\frac{e}{2}+\frac{f\,x}{2}\right)^3+160\tan\left(\frac{e}{2}+\frac{f\,x}{2}\right)^4-480\tan\left(\frac{e}{2}+\right.\right.}{}$$

input
```
int(sin(e + f*x)^2*(a + a*sin(e + f*x))^2*(c - c*sin(e + f*x)),x)
```

output
```
(a^2*c*(15*e - 30*tan(e/2 + (f*x)/2) - 160*tan(e/2 + (f*x)/2)^2 + 180*tan(
e/2 + (f*x)/2)^3 + 160*tan(e/2 + (f*x)/2)^4 - 480*tan(e/2 + (f*x)/2)^6 - 1
80*tan(e/2 + (f*x)/2)^7 + 30*tan(e/2 + (f*x)/2)^9 + 15*f*x + 75*tan(e/2 +
(f*x)/2)^2*(e + f*x) + 150*tan(e/2 + (f*x)/2)^4*(e + f*x) + 150*tan(e/2 +
(f*x)/2)^6*(e + f*x) + 75*tan(e/2 + (f*x)/2)^8*(e + f*x) + 15*tan(e/2 + (f
*x)/2)^10*(e + f*x) - 32))/(120*f*(tan(e/2 + (f*x)/2)^2 + 1)^5)
```

3.3 $\int \sin(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$

3.3.1 Optimal result

Integrand size = 30, antiderivative size = 77

$$\int \sin(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx = \frac{1}{8}a^2cx-\frac{a^2c\cos^3(e+fx)}{3f}-\frac{a^2c\cos(e+fx)\sin(e+fx)}{8f}+\frac{a^2c\cos(e+fx)\sin^3(e+fx)}{4f}$$

output
```
1/8*a^2*c*x-1/3*a^2*c*cos(f*x+e)^3/f-1/8*a^2*c*cos(f*x+e)*sin(f*x+e)/f+1/4
*a^2*c*cos(f*x+e)*sin(f*x+e)^3/f
```

3.3.2 Mathematica [A] (verified)

Time = 0.11 (sec) , antiderivative size = 47, normalized size of antiderivative = 0.61

$$\int \sin(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx = \frac{a^2c(12e+12fx-24\cos(e+fx)-8\cos(3(e+fx))-3\sin(4(e+fx)))}{96f}$$

input
```
Integrate[Sin[e + f*x]*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output
```
(a^2*c*(12*e + 12*f*x - 24*Cos[e + f*x] - 8*Cos[3*(e + f*x)] - 3*Sin[4*(e
+ f*x)]))/(96*f)
```

3.3.3 Rubi [A] (verified)

Time = 0.29 (sec) , antiderivative size = 77, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.100$, Rules used = {3042, 3445, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \sin(e+fx)(a\sin(e+fx)+a)^2(c-c\sin(e+fx))\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int \sin(e+fx)(a\sin(e+fx)+a)^2(c-c\sin(e+fx))dx$$

$$\Big\downarrow \text{3445}$$

$$\int \left(-a^2c\sin^4(e+fx)-a^2c\sin^3(e+fx)+a^2c\sin^2(e+fx)+a^2c\sin(e+fx)\right)dx$$

$$\Big\downarrow \text{2009}$$

$$-\frac{a^2c\cos^3(e+fx)}{3f}+\frac{a^2c\sin^3(e+fx)\cos(e+fx)}{4f}-\frac{a^2c\sin(e+fx)\cos(e+fx)}{8f}+\frac{1}{8}a^2cx$$

input
```
Int[Sin[e + f*x]*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output
```
(a^2*c*x)/8 - (a^2*c*Cos[e + f*x]^3)/(3*f) - (a^2*c*Cos[e + f*x]*Sin[e + f
*x])/(8*f) + (a^2*c*Cos[e + f*x]*Sin[e + f*x]^3)/(4*f)
```

3.3.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3445
```
Int[sin[(e_.) + (f_.)*(x_)]^(n_.)*((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m
_.)*((A_.) + (B_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] :> Int[ExpandTrig[si
n[e + f*x]^n*(a + b*sin[e + f*x])^m*(A + B*sin[e + f*x]), x], x] /; FreeQ[{
a, b, e, f, A, B}, x] && EqQ[A*b + a*B, 0] && EqQ[a^2 - b^2, 0] && IntegerQ
[m] && IntegerQ[n]
```

3.3.4 Maple [A] (verified)

Time = 0.79 (sec) , antiderivative size = 46, normalized size of antiderivative = 0.60

method	result
parallelrisch	$-\frac{a^{2} c \left(-12 f x +24 \cos \left(f x +e \right)+3 \sin \left(4 f x +4 e \right)+8 \cos \left(3 f x +3 e \right)+32\right)}{96 f}$
risch	$\frac{a^{2} c x}{8}-\frac{a^{2} c \cos \left(f x +e \right)}{4 f}-\frac{a^{2} c \sin \left(4 f x +4 e \right)}{32 f}-\frac{a^{2} c \cos \left(3 f x +3 e \right)}{12 f}$
derivativedivides	$\frac{-a^{2} c \left(-\frac{\left(\sin^{3}\left(f x +e \right)+\frac{3 \sin \left(f x +e \right)}{2}\right) \cos \left(f x +e \right)}{4}+\frac{3 f x}{8}+\frac{3 e}{8}\right)+\frac{a^{2} c \left(2+\sin^{2}\left(f x +e \right)\right) \cos \left(f x +e \right)}{3}+a^{2} c \left(-\frac{\cos \left(f x +e \right) \sin \left(f x +e \right)}{2}+ \right.}{f}$
default	$\frac{-a^{2} c \left(-\frac{\left(\sin^{3}\left(f x +e \right)+\frac{3 \sin \left(f x +e \right)}{2}\right) \cos \left(f x +e \right)}{4}+\frac{3 f x}{8}+\frac{3 e}{8}\right)+\frac{a^{2} c \left(2+\sin^{2}\left(f x +e \right)\right) \cos \left(f x +e \right)}{3}+a^{2} c \left(-\frac{\cos \left(f x +e \right) \sin \left(f x +e \right)}{2}+ \right.}{f}$
parts	$-\frac{a^{2} c \cos \left(f x +e \right)}{f}+\frac{a^{2} c \left(-\frac{\cos \left(f x +e \right) \sin \left(f x +e \right)}{2}+\frac{f x}{2}+\frac{e}{2}\right)}{f}+\frac{a^{2} c \left(2+\sin^{2}\left(f x +e \right)\right) \cos \left(f x +e \right)}{3 f}-\frac{a^{2} c \left(-\frac{\left(\sin^{3}\left(f x +e \right)+ \right.}{} \right.}{}$
norman	$\frac{-\frac{2 a^{2} c}{3 f}+\frac{a^{2} c x}{8}-\frac{2 a^{2} c \left(\tan^{2}\left(\frac{f x}{2}+\frac{e}{2}\right)\right)}{3 f}-\frac{2 a^{2} c \left(\tan^{4}\left(\frac{f x}{2}+\frac{e}{2}\right)\right)}{f}-\frac{2 a^{2} c \left(\tan^{6}\left(\frac{f x}{2}+\frac{e}{2}\right)\right)}{f}-\frac{a^{2} c \tan \left(\frac{f x}{2}+\frac{e}{2}\right)}{4 f}+\frac{7 a^{2} c \left(\tan^{3}\left(\frac{f x}{2}+\frac{e}{2}\right)\right)}{4 f}-}{\left(1+\mathrm{t} \right.}$

input
```
int(sin(f*x+e)*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x,method=_RETURNVERBOSE
)
```

output
```
-1/96*a^2*c*(-12*f*x+24*cos(f*x+e)+3*sin(4*f*x+4*e)+8*cos(3*f*x+3*e)+32)/f
```

3.3.5 Fricas [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 63, normalized size of antiderivative = 0.82

$$\int \sin(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$= -\frac{8\,a^2c\cos\left(fx+e\right)^3 - 3\,a^2cfx + 3\left(2\,a^2c\cos\left(fx+e\right)^3 - a^2c\cos\left(fx+e\right)\right)\sin\left(fx+e\right)}{24\,f}$$

input
```
integrate(sin(f*x+e)*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="fri
cas")
```

output
```
-1/24*(8*a^2*c*cos(f*x + e)^3 - 3*a^2*c*f*x + 3*(2*a^2*c*cos(f*x + e)^3 -
a^2*c*cos(f*x + e))*sin(f*x + e))/f
```

3.3.6 Sympy [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 245 vs. 2(70) = 140.

Time = 0.18 (sec) , antiderivative size = 245, normalized size of antiderivative = 3.18

$$\int \sin(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$= \begin{cases} -\frac{3a^2cx\sin^4(e+fx)}{8} - \frac{3a^2cx\sin^2(e+fx)\cos^2(e+fx)}{4} + \frac{a^2cx\sin^2(e+fx)}{2} - \frac{3a^2cx\cos^4(e+fx)}{8} + \frac{a^2cx\cos^2(e+fx)}{2} + \frac{5a^2c\sin^3(e+ }{8} \\ x(a\sin(e)+a)^2(-c\sin(e)+c)\sin(e) \end{cases}$$

input
```
integrate(sin(f*x+e)*(a+a*sin(f*x+e))**2*(c-c*sin(f*x+e)),x)
```

output
```
Piecewise((-3*a**2*c*x*sin(e + f*x)**4/8 - 3*a**2*c*x*sin(e + f*x)**2*cos(
e + f*x)**2/4 + a**2*c*x*sin(e + f*x)**2/2 - 3*a**2*c*x*cos(e + f*x)**4/8
+ a**2*c*x*cos(e + f*x)**2/2 + 5*a**2*c*sin(e + f*x)**3*cos(e + f*x)/(8*f)
 + a**2*c*sin(e + f*x)**2*cos(e + f*x)/f + 3*a**2*c*sin(e + f*x)*cos(e + f
*x)**3/(8*f) - a**2*c*sin(e + f*x)*cos(e + f*x)/(2*f) + 2*a**2*c*cos(e + f
*x)**3/(3*f) - a**2*c*cos(e + f*x)/f, Ne(f, 0)), (x*(a*sin(e) + a)**2*(-c*
sin(e) + c)*sin(e), True))
```

3.3.7 Maxima [A] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 100, normalized size of antiderivative = 1.30

$$\int \sin(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx =$$
$$-\frac{32\left(\cos\left(fx+e\right)^3-3\,\cos\left(fx+e\right)\right)a^2c+3\left(12\,fx+12\,e+\sin\left(4\,fx+4\,e\right)-8\,\sin\left(2\,fx+2\,e\right)\right)a^2c-2}{96\,f}$$

input
```
integrate(sin(f*x+e)*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="max
ima")
```

output
```
-1/96*(32*(cos(f*x + e)^3 - 3*cos(f*x + e))*a^2*c + 3*(12*f*x + 12*e + sin
(4*f*x + 4*e) - 8*sin(2*f*x + 2*e))*a^2*c - 24*(2*f*x + 2*e - sin(2*f*x +
2*e))*a^2*c + 96*a^2*c*cos(f*x + e))/f
```

3.3.8 Giac [A] (verification not implemented)

Time = 0.37 (sec) , antiderivative size = 59, normalized size of antiderivative = 0.77

$$\int \sin(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$=\frac{1}{8}\,a^2cx-\frac{a^2c\cos\left(3\,fx+3\,e\right)}{12\,f}-\frac{a^2c\cos\left(fx+e\right)}{4\,f}-\frac{a^2c\sin\left(4\,fx+4\,e\right)}{32\,f}$$

input
```
integrate(sin(f*x+e)*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="gia
c")
```

output
```
1/8*a^2*c*x - 1/12*a^2*c*cos(3*f*x + 3*e)/f - 1/4*a^2*c*cos(f*x + e)/f - 1
/32*a^2*c*sin(4*f*x + 4*e)/f
```

3.3.9 Mupad [B] (verification not implemented)

Time = 14.06 (sec) , antiderivative size = 250, normalized size of antiderivative = 3.25

$$\int \sin(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx = \frac{a^2\,c\,x}{8}$$

$$-\frac{\tan\left(\frac{e}{2}+\frac{f\,x}{2}\right)^2\left(\frac{a^2\,c\,(3\,e+3\,f\,x)}{6}-\frac{a^2\,c\,(12\,e+12\,f\,x-16)}{24}\right)+\tan\left(\frac{e}{2}+\frac{f\,x}{2}\right)^6\left(\frac{a^2\,c\,(3\,e+3\,f\,x)}{6}-\frac{a^2\,c\,(12\,e+12\,f\,x-48)}{24}\right)+\mathrm{ta}}{}$$

input
```
int(sin(e + f*x)*(a + a*sin(e + f*x))^2*(c - c*sin(e + f*x)),x)
```

output
```
(a^2*c*x)/8 - (tan(e/2 + (f*x)/2)^2*((a^2*c*(3*e + 3*f*x))/6 - (a^2*c*(12*
e + 12*f*x - 16))/24) + tan(e/2 + (f*x)/2)^6*((a^2*c*(3*e + 3*f*x))/6 - (a
^2*c*(12*e + 12*f*x - 48))/24) + tan(e/2 + (f*x)/2)^4*((a^2*c*(3*e + 3*f*x
))/4 - (a^2*c*(18*e + 18*f*x - 48))/24) + (a^2*c*tan(e/2 + (f*x)/2))/4 - (
7*a^2*c*tan(e/2 + (f*x)/2)^3)/4 + (7*a^2*c*tan(e/2 + (f*x)/2)^5)/4 - (a^2*
c*tan(e/2 + (f*x)/2)^7)/4 + (a^2*c*(3*e + 3*f*x))/24 - (a^2*c*(3*e + 3*f*x
 - 16))/24)/(f*(tan(e/2 + (f*x)/2)^2 + 1)^4)
```

3.4 $\int (a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$

3.4.1 Optimal result

Integrand size = 24, antiderivative size = 52

$$\int (a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx = \frac{1}{2}a^2cx - \frac{a^2c\cos^3(e+fx)}{3f} + \frac{a^2c\cos(e+fx)\sin(e+fx)}{2f}$$

output
```
1/2*a^2*c*x-1/3*a^2*c*cos(f*x+e)^3/f+1/2*a^2*c*cos(f*x+e)*sin(f*x+e)/f
```

3.4.2 Mathematica [A] (verified)

Time = 0.58 (sec) , antiderivative size = 43, normalized size of antiderivative = 0.83

$$\int (a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx = -\frac{a^2c(3\cos(e+fx)+\cos(3(e+fx))-3(2fx+\sin(2(e+fx))))}{12f}$$

input
```
Integrate[(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output
```
-1/12*(a^2*c*(3*Cos[e + f*x] + Cos[3*(e + f*x)] - 3*(2*f*x + Sin[2*(e + f*
x)])))/f
```

3.4.3 Rubi [A] (verified)

Time = 0.31 (sec) , antiderivative size = 47, normalized size of antiderivative = 0.90, number of steps used = 7, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.292, Rules used = {3042, 3215, 3042, 3148, 3042, 3115, 24}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int (a\sin(e+fx)+a)^2(c-c\sin(e+fx))\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int (a\sin(e+fx)+a)^2(c-c\sin(e+fx))dx$$

$$\Big\downarrow \text{3215}$$

$$ac\int \cos^2(e+fx)(\sin(e+fx)a+a)dx$$

$$\Big\downarrow \text{3042}$$

$$ac\int \cos(e+fx)^2(\sin(e+fx)a+a)dx$$

$$\Big\downarrow \text{3148}$$

$$ac\left(a\int \cos^2(e+fx)dx-\frac{a\cos^3(e+fx)}{3f}\right)$$

$$\Big\downarrow \text{3042}$$

$$ac\left(a\int \sin\left(e+fx+\frac{\pi}{2}\right)^2dx-\frac{a\cos^3(e+fx)}{3f}\right)$$

$$\Big\downarrow \text{3115}$$

$$ac\left(a\left(\frac{\int 1dx}{2}+\frac{\sin(e+fx)\cos(e+fx)}{2f}\right)-\frac{a\cos^3(e+fx)}{3f}\right)$$

$$\Big\downarrow \text{24}$$

$$ac\left(a\left(\frac{\sin(e+fx)\cos(e+fx)}{2f}+\frac{x}{2}\right)-\frac{a\cos^3(e+fx)}{3f}\right)$$

input
```
Int[(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output
```
a*c*(-1/3*(a*Cos[e + f*x]^3)/f + a*(x/2 + (Cos[e + f*x]*Sin[e + f*x])/(2*f
)))
```

3.4.3.1 Defintions of rubi rules used

rule 24
```
Int[a_, x_Symbol] :> Simp[a*x, x] /; FreeQ[a, x]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3115
```
Int[((b_.)*sin[(c_.) + (d_.)*(x_)])^(n_), x_Symbol] :> Simp[(-b)*Cos[c + d*
x]*((b*Sin[c + d*x])^(n - 1)/(d*n)), x] + Simp[b^2*((n - 1)/n)   Int[(b*Sin
[c + d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[
2*n]
```

rule 3148
```
Int[(cos[(e_.) + (f_.)*(x_)]*(g_.))^(p_)*((a_) + (b_.)*sin[(e_.) + (f_.)*(x
_)]), x_Symbol] :> Simp[(-b)*((g*Cos[e + f*x])^(p + 1)/(f*g*(p + 1))), x] +
 Simp[a   Int[(g*Cos[e + f*x])^p, x], x] /; FreeQ[{a, b, e, f, g, p}, x] &&
 (IntegerQ[2*p] || NeQ[a^2 - b^2, 0])
```

rule 3215
```
Int[((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_.)*((c_) + (d_.)*sin[(e_.) +
(f_.)*(x_)])^(n_.), x_Symbol] :> Simp[a^m*c^m   Int[Cos[e + f*x]^(2*m)*(c +
 d*Sin[e + f*x])^(n - m), x], x] /; FreeQ[{a, b, c, d, e, f, n}, x] && EqQ[
b*c + a*d, 0] && EqQ[a^2 - b^2, 0] && IntegerQ[m] &&  !(IntegerQ[n] && ((Lt
Q[m, 0] && GtQ[n, 0]) || LtQ[0, n, m] || LtQ[m, n, 0]))
```

3.4.4 Maple [A] (verified)

Time = 0.60 (sec) , antiderivative size = 44, normalized size of antiderivative = 0.85

method	result
parallelrisch	$-\frac{a^{2} c \left(-6 f x +3 \cos \left(f x +e \right)+\cos \left(3 f x +3 e \right)-3 \sin \left(2 f x +2 e \right)+4\right)}{12 f}$
risch	$\frac{a^{2} c x}{2}-\frac{a^{2} c \cos \left(f x +e \right)}{4 f}-\frac{a^{2} c \cos \left(3 f x +3 e \right)}{12 f}+\frac{a^{2} c \sin \left(2 f x +2 e \right)}{4 f}$
derivativedivides	$\frac{\frac{a^{2} c \left(2+\sin^{2}\left(f x +e \right)\right) \cos \left(f x +e \right)}{3}-a^{2} c \left(-\frac{\cos \left(f x +e \right) \sin \left(f x +e \right)}{2}+\frac{f x}{2}+\frac{e}{2}\right)-a^{2} c \cos \left(f x +e \right)+a^{2} c \left(f x +e \right)}{f}$
default	$\frac{\frac{a^{2} c \left(2+\sin^{2}\left(f x +e \right)\right) \cos \left(f x +e \right)}{3}-a^{2} c \left(-\frac{\cos \left(f x +e \right) \sin \left(f x +e \right)}{2}+\frac{f x}{2}+\frac{e}{2}\right)-a^{2} c \cos \left(f x +e \right)+a^{2} c \left(f x +e \right)}{f}$
parts	$a^{2} c x -\frac{a^{2} c \cos \left(f x +e \right)}{f}-\frac{a^{2} c \left(-\frac{\cos \left(f x +e \right) \sin \left(f x +e \right)}{2}+\frac{f x}{2}+\frac{e}{2}\right)}{f}+\frac{a^{2} c \left(2+\sin^{2}\left(f x +e \right)\right) \cos \left(f x +e \right)}{3 f}$
norman	$\frac{\frac{a^{2} c \tan \left(\frac{f x}{2}+\frac{e}{2}\right)}{f}-\frac{2 a^{2} c}{3 f}+\frac{a^{2} c x}{2}-\frac{2 a^{2} c \left(\tan^{4}\left(\frac{f x}{2}+\frac{e}{2}\right)\right)}{f}-\frac{a^{2} c \left(\tan^{5}\left(\frac{f x}{2}+\frac{e}{2}\right)\right)}{f}+\frac{3 a^{2} c x \left(\tan^{2}\left(\frac{f x}{2}+\frac{e}{2}\right)\right)}{2}+\frac{3 a^{2} c x \left(\tan^{4}\left(\frac{f x}{2}+\frac{e}{2}\right)\right)}{2}+}{\left(1+\tan^{2}\left(\frac{f x}{2}+\frac{e}{2}\right)\right)^{3}}$

input
```
int((a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x,method=_RETURNVERBOSE)
```

output
```
-1/12*a^2*c*(-6*f*x+3*cos(f*x+e)+cos(3*f*x+3*e)-3*sin(2*f*x+2*e)+4)/f
```

3.4.5 Fricas [A] (verification not implemented)

Time = 0.26 (sec) , antiderivative size = 46, normalized size of antiderivative = 0.88

$$\int (a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx = -\frac{2\,a^{2} c \cos\left(fx+e\right)^{3} - 3\,a^{2} c f x - 3\,a^{2} c \cos\left(fx+e\right) \sin\left(fx+e\right)}{6\,f}$$

input
```
integrate((a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="fricas")
```

output
```
-1/6*(2*a^2*c*cos(f*x + e)^3 - 3*a^2*c*f*x - 3*a^2*c*cos(f*x + e)*sin(f*x
+ e))/f
```

3.4.6 Sympy [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 133 vs. 2(46) = 92.

Time = 0.13 (sec) , antiderivative size = 133, normalized size of antiderivative = 2.56

$$\int (a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$= \begin{cases} -\frac{a^2cx\sin^2(e+fx)}{2} - \frac{a^2cx\cos^2(e+fx)}{2} + a^2cx + \frac{a^2c\sin^2(e+fx)\cos(e+fx)}{f} + \frac{a^2c\sin(e+fx)\cos(e+fx)}{2f} + \frac{2a^2c\cos^3(e+fx)}{3f} - \\ x(a\sin(e)+a)^2(-c\sin(e)+c) \end{cases}$$

input
```
integrate((a+a*sin(f*x+e))**2*(c-c*sin(f*x+e)),x)
```

output
```
Piecewise((-a**2*c*x*sin(e + f*x)**2/2 - a**2*c*x*cos(e + f*x)**2/2 + a**2
*c*x + a**2*c*sin(e + f*x)**2*cos(e + f*x)/f + a**2*c*sin(e + f*x)*cos(e +
 f*x)/(2*f) + 2*a**2*c*cos(e + f*x)**3/(3*f) - a**2*c*cos(e + f*x)/f, Ne(f
, 0)), (x*(a*sin(e) + a)**2*(-c*sin(e) + c), True))
```

3.4.7 Maxima [A] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 77, normalized size of antiderivative = 1.48

$$\int (a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx =$$
$$-\frac{4\left(\cos(fx+e)^3 - 3\cos(fx+e)\right)a^2c + 3\left(2fx+2e-\sin(2fx+2e)\right)a^2c - 12(fx+e)a^2c + 12a^2c\,c}{12f}$$

input
```
integrate((a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="maxima")
```

output
```
-1/12*(4*(cos(f*x + e)^3 - 3*cos(f*x + e))*a^2*c + 3*(2*f*x + 2*e - sin(2*
f*x + 2*e))*a^2*c - 12*(f*x + e)*a^2*c + 12*a^2*c*cos(f*x + e))/f
```

3.4.8 Giac [A] (verification not implemented)

Time = 0.59 (sec) , antiderivative size = 59, normalized size of antiderivative = 1.13

$$\int (a + a\sin(e + fx))^2(c - c\sin(e + fx))\,dx = \frac{1}{2}\,a^2cx - \frac{a^2c\cos\left(3\,fx + 3\,e\right)}{12\,f} - \frac{a^2c\cos\left(fx + e\right)}{4\,f} + \frac{a^2c\sin\left(2\,fx + 2\,e\right)}{4\,f}$$

input
```
integrate((a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="giac")
```

output
```
1/2*a^2*c*x - 1/12*a^2*c*cos(3*f*x + 3*e)/f - 1/4*a^2*c*cos(f*x + e)/f + 1
/4*a^2*c*sin(2*f*x + 2*e)/f
```

3.4.9 Mupad [B] (verification not implemented)

Time = 14.52 (sec) , antiderivative size = 125, normalized size of antiderivative = 2.40

$$\int (a + a\sin(e + fx))^2(c - c\sin(e + fx))\,dx = \frac{a^2\,c\,x}{2} - \frac{\tan\left(\frac{e}{2} + \frac{f\,x}{2}\right)^4\left(\frac{3\,a^2\,c\,(e+f\,x)}{2} - \frac{a^2\,c\,(9\,e+9\,f\,x-12)}{6}\right) - a^2\,c\tan\left(\frac{e}{2} + \frac{f\,x}{2}\right) + \frac{a^2\,c\,(e+f\,x)}{2} + a^2\,c\tan\left(\frac{e}{2} + \frac{f\,x}{2}\right)^5 - a^2\,c}{f\left(\tan\left(\frac{e}{2} + \frac{f\,x}{2}\right)^2 + 1\right)^3}$$

input
```
int((a + a*sin(e + f*x))^2*(c - c*sin(e + f*x)),x)
```

output
```
(a^2*c*x)/2 - (tan(e/2 + (f*x)/2)^4*((3*a^2*c*(e + f*x))/2 - (a^2*c*(9*e +
 9*f*x - 12))/6) - a^2*c*tan(e/2 + (f*x)/2) + (a^2*c*(e + f*x))/2 + a^2*c*
tan(e/2 + (f*x)/2)^5 - (a^2*c*(3*e + 3*f*x - 4))/6)/(f*(tan(e/2 + (f*x)/2)
^2 + 1)^3)
```

3.5 $\int \csc(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$

3.5.1 Optimal result

Integrand size = 30, antiderivative size = 63

$$\int \csc(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx = \frac{1}{2}a^2cx-\frac{a^2c\,\text{arctanh}(\cos(e+fx))}{f}+\frac{a^2c\cos(e+fx)}{f}+\frac{a^2c\cos(e+fx)\sin(e+fx)}{2f}$$

output
```
1/2*a^2*c*x-a^2*c*arctanh(cos(f*x+e))/f+a^2*c*cos(f*x+e)/f+1/2*a^2*c*cos(f
*x+e)*sin(f*x+e)/f
```

3.5.2 Mathematica [A] (verified)

Time = 0.14 (sec) , antiderivative size = 61, normalized size of antiderivative = 0.97

$$\int \csc(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx = \frac{a^2c\left(-2e+2fx+4\cos(e+fx)-4\log\left(\cos\left(\frac{1}{2}(e+fx)\right)\right)+4\log\left(\sin\left(\frac{1}{2}(e+fx)\right)\right)+\sin(2(e+fx))\right)}{4f}$$

input
```
Integrate[Csc[e + f*x]*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output
```
(a^2*c*(-2*e + 2*f*x + 4*Cos[e + f*x] - 4*Log[Cos[(e + f*x)/2]] + 4*Log[Si
n[(e + f*x)/2]] + Sin[2*(e + f*x)]))/(4*f)
```

3.5.3 Rubi [A] (verified)

Time = 0.29 (sec) , antiderivative size = 63, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.100$, Rules used = {3042, 3445, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \csc(e+fx)(a\sin(e+fx)+a)^2(c-c\sin(e+fx))\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{(a\sin(e+fx)+a)^2(c-c\sin(e+fx))}{\sin(e+fx)}dx$$

$$\Big\downarrow \text{3445}$$

$$\int \left(a^2(-c)\sin^2(e+fx)-a^2c\sin(e+fx)+a^2c\csc(e+fx)+a^2c\right)dx$$

$$\Big\downarrow \text{2009}$$

$$-\frac{a^2c\operatorname{arctanh}(\cos(e+fx))}{f}+\frac{a^2c\cos(e+fx)}{f}+\frac{a^2c\sin(e+fx)\cos(e+fx)}{2f}+\frac{1}{2}a^2cx$$

input
```
Int[Csc[e + f*x]*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output
```
(a^2*c*x)/2 - (a^2*c*ArcTanh[Cos[e + f*x]])/f + (a^2*c*Cos[e + f*x])/f + (
a^2*c*Cos[e + f*x]*Sin[e + f*x])/(2*f)
```

3.5.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3445
```
Int[sin[(e_.) + (f_.)*(x_)]^(n_.)*((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m
_.)*((A_.) + (B_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] :> Int[ExpandTrig[si
n[e + f*x]^n*(a + b*sin[e + f*x])^m*(A + B*sin[e + f*x]), x], x] /; FreeQ[{
a, b, e, f, A, B}, x] && EqQ[A*b + a*B, 0] && EqQ[a^2 - b^2, 0] && IntegerQ
[m] && IntegerQ[n]
```

3.5.4 Maple [A] (verified)

Time = 0.46 (sec) , antiderivative size = 45, normalized size of antiderivative = 0.71

method	result
parallelrisch	$\frac{a^{2} c \left(2 f x+4 \cos(f x+e)+4 \ln\left(\tan\left(\frac{f x}{2}+\frac{e}{2}\right)\right)+\sin(2 f x+2 e)+4\right)}{4 f}$
derivativedivides	$\frac{-a^{2} c \left(-\frac{\cos(f x+e) \sin(f x+e)}{2}+\frac{f x}{2}+\frac{e}{2}\right)+a^{2} c \cos(f x+e)+a^{2} c (f x+e)+a^{2} c \ln(\csc(f x+e)-\cot(f x+e))}{f}$
default	$\frac{-a^{2} c \left(-\frac{\cos(f x+e) \sin(f x+e)}{2}+\frac{f x}{2}+\frac{e}{2}\right)+a^{2} c \cos(f x+e)+a^{2} c (f x+e)+a^{2} c \ln(\csc(f x+e)-\cot(f x+e))}{f}$
risch	$\frac{a^{2} c x}{2}+\frac{a^{2} c \,{\mathrm e}^{i (f x+e)}}{2 f}+\frac{a^{2} c \,{\mathrm e}^{-i (f x+e)}}{2 f}-\frac{a^{2} c \ln\left({\mathrm e}^{i (f x+e)}+1\right)}{f}+\frac{a^{2} c \ln\left({\mathrm e}^{i (f x+e)}-1\right)}{f}+\frac{a^{2} c \sin(2 f x+2 e)}{4 f}$
norman	$\frac{\frac{a^{2} c \tan\left(\frac{f x}{2}+\frac{e}{2}\right)}{f}+\frac{a^{2} c x}{2}-\frac{2 a^{2} c \left(\tan^{2}\left(\frac{f x}{2}+\frac{e}{2}\right)\right)}{f}-\frac{2 a^{2} c \left(\tan^{6}\left(\frac{f x}{2}+\frac{e}{2}\right)\right)}{f}-\frac{4 a^{2} c \left(\tan^{4}\left(\frac{f x}{2}+\frac{e}{2}\right)\right)}{f}-\frac{a^{2} c \left(\tan^{5}\left(\frac{f x}{2}+\frac{e}{2}\right)\right)}{f}+\frac{3 a^{2} c x (\tan}{}}{\left(1+\tan^{2}\left(\frac{f x}{2}+\frac{e}{2}\right)\right)^{3}}$

input
```
int(csc(f*x+e)*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x,method=_RETURNVERBOSE
)
```

output
```
1/4*a^2*c*(2*f*x+4*cos(f*x+e)+4*ln(tan(1/2*f*x+1/2*e))+sin(2*f*x+2*e)+4)/f
```

3.5.5 Fricas [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 75, normalized size of antiderivative = 1.19

$$\int \csc(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$=\frac{a^2cfx+a^2c\cos(fx+e)\sin(fx+e)+2\,a^2c\cos(fx+e)-a^2c\log\left(\frac{1}{2}\cos(fx+e)+\frac{1}{2}\right)+a^2c\log\left(-\frac{1}{2}\,\mathrm{co}\right.}{2\,f}$$

input
```
integrate(csc(f*x+e)*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="fri
cas")
```

output
```
1/2*(a^2*c*f*x + a^2*c*cos(f*x + e)*sin(f*x + e) + 2*a^2*c*cos(f*x + e) -
a^2*c*log(1/2*cos(f*x + e) + 1/2) + a^2*c*log(-1/2*cos(f*x + e) + 1/2))/f
```

3.5.6 Sympy [F]

$$\int \csc(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$= -a^2c\left(\int (-\sin(e+fx)\csc(e+fx))\;dx + \int \sin^2(e+fx)\csc(e+fx)\,dx + \int \sin^3(e+fx)\csc(e+fx)\,dx + \int (-\csc(e+fx))\;dx\right)$$

input
```
integrate(csc(f*x+e)*(a+a*sin(f*x+e))**2*(c-c*sin(f*x+e)),x)
```

output
```
-a**2*c*(Integral(-sin(e + f*x)*csc(e + f*x), x) + Integral(sin(e + f*x)**
2*csc(e + f*x), x) + Integral(sin(e + f*x)**3*csc(e + f*x), x) + Integral(
-csc(e + f*x), x))
```

3.5.7 Maxima [A] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 73, normalized size of antiderivative = 1.16

$$\int \csc(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx =$$
$$-\frac{(2\,fx+2\,e-\sin(2\,fx+2\,e))a^2c-4\,(fx+e)a^2c-4\,a^2c\cos(fx+e)+4\,a^2c\log(\cot(fx+e)+\csc(f$$
$$4\,f$$

input
```
integrate(csc(f*x+e)*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="max
ima")
```

output
```
-1/4*((2*f*x + 2*e - sin(2*f*x + 2*e))*a^2*c - 4*(f*x + e)*a^2*c - 4*a^2*c
*cos(f*x + e) + 4*a^2*c*log(cot(f*x + e) + csc(f*x + e)))/f
```

3.5.8 Giac [A] (verification not implemented)

Time = 0.33 (sec) , antiderivative size = 105, normalized size of antiderivative = 1.67

$$\int \csc(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$=\frac{(fx+e)a^2c+2\,a^2c\log\left(\left|\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right)\right|\right)-\frac{2\left(a^2c\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right)^3-2\,a^2c\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right)^2-a^2c\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right)-2\,a^2c\right)}{\left(\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right)^2+1\right)^2}}{2\,f}$$

input
```
integrate(csc(f*x+e)*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="gia
c")
```

output
```
1/2*((f*x + e)*a^2*c + 2*a^2*c*log(abs(tan(1/2*f*x + 1/2*e))) - 2*(a^2*c*t
an(1/2*f*x + 1/2*e)^3 - 2*a^2*c*tan(1/2*f*x + 1/2*e)^2 - a^2*c*tan(1/2*f*x
 + 1/2*e) - 2*a^2*c)/(tan(1/2*f*x + 1/2*e)^2 + 1)^2)/f
```

3.5.9 Mupad [B] (verification not implemented)

Time = 12.67 (sec) , antiderivative size = 88, normalized size of antiderivative = 1.40

$$\int \csc(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$=\frac{a^2\,c\left(\cos\left(e+f\,x\right)+\ln\left(\frac{\sin\left(\frac{e}{2}+\frac{f\,x}{2}\right)}{\cos\left(\frac{e}{2}+\frac{f\,x}{2}\right)}\right)+\frac{\sin(2\,e+2\,f\,x)}{4}+\mathrm{atan}\left(\frac{\sqrt{5}\left(\cos\left(\frac{e}{2}+\frac{f\,x}{2}\right)+2\,\sin\left(\frac{e}{2}+\frac{f\,x}{2}\right)\right)}{5\,\cos\left(\frac{e}{2}+\mathrm{atan}\left(\frac{1}{2}\right)+\frac{f\,x}{2}\right)}\right)\right)}{f}$$

input
```
int(((a + a*sin(e + f*x))^2*(c - c*sin(e + f*x)))/sin(e + f*x),x)
```

output
```
(a^2*c*(cos(e + f*x) + log(sin(e/2 + (f*x)/2)/cos(e/2 + (f*x)/2)) + sin(2*
e + 2*f*x)/4 + atan((5^(1/2)*(cos(e/2 + (f*x)/2) + 2*sin(e/2 + (f*x)/2)))/
(5*cos(e/2 + atan(1/2) + (f*x)/2)))))/f
```

3.6 $\int \csc^2(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$

3.6.1 Optimal result

Integrand size = 32, antiderivative size = 53

$$\int \csc^2(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx = -a^2cx-\frac{a^2c\,\text{arctanh}(\cos(e+fx))}{f}+\frac{a^2c\cos(e+fx)}{f}-\frac{a^2c\cot(e+fx)}{f}$$

output
```
-a^2*c*x-a^2*c*arctanh(cos(f*x+e))/f+a^2*c*cos(f*x+e)/f-a^2*c*cot(f*x+e)/f
```

3.6.2 Mathematica [A] (verified)

Time = 0.08 (sec) , antiderivative size = 97, normalized size of antiderivative = 1.83

$$\int \csc^2(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx = -a^2cx+\frac{a^2c\cos(e)\cos(fx)}{f}-\frac{a^2c\cot(e+fx)}{f}-\frac{a^2c\log\left(\cos\left(\frac{e}{2}+\frac{fx}{2}\right)\right)}{f}+\frac{a^2c\log\left(\sin\left(\frac{e}{2}+\frac{fx}{2}\right)\right)}{f}-\frac{a^2c\sin(e)\sin(fx)}{f}$$

input
```
Integrate[Csc[e + f*x]^2*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output
```
-(a^2*c*x) + (a^2*c*Cos[e]*Cos[f*x])/f - (a^2*c*Cot[e + f*x])/f - (a^2*c*L
og[Cos[e/2 + (f*x)/2]])/f + (a^2*c*Log[Sin[e/2 + (f*x)/2]])/f - (a^2*c*Sin
[e]*Sin[f*x])/f
```

3.6.3 Rubi [A] (verified)

Time = 0.36 (sec) , antiderivative size = 44, normalized size of antiderivative = 0.83, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.156$, Rules used = {3042, 3429, 3042, 3189, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \csc^2(e+fx)(a\sin(e+fx)+a)^2(c-c\sin(e+fx))\,dx$$

$$\Big\downarrow \; 3042$$

$$\int \frac{(a\sin(e+fx)+a)^2(c-c\sin(e+fx))}{\sin(e+fx)^2}dx$$

$$\Big\downarrow \; 3429$$

$$ac\int \cot^2(e+fx)(\sin(e+fx)a+a)dx$$

$$\Big\downarrow \; 3042$$

$$ac\int \frac{\sin(e+fx)a+a}{\tan(e+fx)^2}dx$$

$$\Big\downarrow \; 3189$$

$$ac\int \left(a\cot^2(e+fx)+a\cos(e+fx)\cot(e+fx)\right)dx$$

$$\Big\downarrow \; 2009$$

$$ac\left(-\frac{a\text{arctanh}(\cos(e+fx))}{f}+\frac{a\cos(e+fx)}{f}-\frac{a\cot(e+fx)}{f}-ax\right)$$

input
```
Int[Csc[e + f*x]^2*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output
```
a*c*(-(a*x) - (a*ArcTanh[Cos[e + f*x]])/f + (a*Cos[e + f*x])/f - (a*Cot[e
+ f*x])/f)
```

3.6.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3189
```
Int[((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_.)*((g_.)*tan[(e_.) + (f_.)*(
x_)])^(p_.), x_Symbol] :> Int[ExpandIntegrand[(g*Tan[e + f*x])^p, (a + b*Si
n[e + f*x])^m, x], x] /; FreeQ[{a, b, e, f, g, p}, x] && EqQ[a^2 - b^2, 0]
&& IGtQ[m, 0]
```

rule 3429
```
Int[sin[(e_.) + (f_.)*(x_)]^(p_)*((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_
.)*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])^(n_.), x_Symbol] :> Simp[a^n*c^n
  Int[Tan[e + f*x]^p*(a + b*Sin[e + f*x])^(m - n), x], x] /; FreeQ[{a, b, c
, d, e, f, m}, x] && EqQ[b*c + a*d, 0] && EqQ[a^2 - b^2, 0] && EqQ[p + 2*n,
 0] && IntegerQ[n]
```

3.6.4 Maple [A] (verified)

Time = 0.53 (sec) , antiderivative size = 61, normalized size of antiderivative = 1.15

method	result
derivativedivides	$\frac{a^{2} c \cos(fx+e)-a^{2} c (fx+e)+a^{2} c \ln(\csc(fx+e)-\cot(fx+e))-a^{2} c \cot(fx+e)}{f}$
default	$\frac{a^{2} c \cos(fx+e)-a^{2} c (fx+e)+a^{2} c \ln(\csc(fx+e)-\cot(fx+e))-a^{2} c \cot(fx+e)}{f}$
parallelrisch	$\frac{a^{2} c \left(-2 f x-2+2 \ln\left(\tan\left(\frac{f x}{2}+\frac{e}{2}\right)\right)+2 \cos(fx+e)+2 \tan\left(\frac{f x}{2}+\frac{e}{2}\right)-\sec\left(\frac{f x}{2}+\frac{e}{2}\right) \csc\left(\frac{f x}{2}+\frac{e}{2}\right)\right)}{2 f}$
risch	$-a^{2} c x+\frac{a^{2} c \,{\mathrm e}^{i(fx+e)}}{2 f}+\frac{a^{2} c \,{\mathrm e}^{-i(fx+e)}}{2 f}-\frac{2 i a^{2} c}{f \left({\mathrm e}^{2 i (fx+e)}-1\right)}-\frac{a^{2} c \ln\left({\mathrm e}^{i(fx+e)}+1\right)}{f}+\frac{a^{2} c \ln\left({\mathrm e}^{i(fx+e)}-1\right)}{f}$
norman	$\frac{\frac{a^{2} c \left(\tan^{6}\left(\frac{f x}{2}+\frac{e}{2}\right)\right)}{f}-\frac{a^{2} c}{2 f}-\frac{2 a^{2} c \left(\tan^{3}\left(\frac{f x}{2}+\frac{e}{2}\right)\right)}{f}-\frac{2 a^{2} c \left(\tan^{7}\left(\frac{f x}{2}+\frac{e}{2}\right)\right)}{f}-\frac{4 a^{2} c \left(\tan^{5}\left(\frac{f x}{2}+\frac{e}{2}\right)\right)}{f}-\frac{a^{2} c \left(\tan^{2}\left(\frac{f x}{2}+\frac{e}{2}\right)\right)}{f}+\frac{a^{2} c (\tan}{}}{\tan\left(\frac{f x}{2}+\frac{e}{2}\right)\left(1+\tan^{2}\left(\right.\right.}$

input
```
int(csc(f*x+e)^2*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x,method=_RETURNVERBO
SE)
```

output
```
1/f*(a^2*c*cos(f*x+e)-a^2*c*(f*x+e)+a^2*c*ln(csc(f*x+e)-cot(f*x+e))-a^2*c*
cot(f*x+e))
```

3.6.5 Fricas [A] (verification not implemented)

Time = 0.28 (sec) , antiderivative size = 99, normalized size of antiderivative = 1.87

$$\int \csc^2(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx = -\frac{a^2c\log\left(\frac{1}{2}\cos\left(fx+e\right)+\frac{1}{2}\right)\sin\left(fx+e\right)-a^2c\log\left(-\frac{1}{2}\cos\left(fx+e\right)+\frac{1}{2}\right)\sin\left(fx+e\right)+2\,a^2c\cos\left(fx+\right.}{2\,f\sin\left(fx+e\right)}$$

input
```
integrate(csc(f*x+e)^2*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="f
ricas")
```

output
```
-1/2*(a^2*c*log(1/2*cos(f*x + e) + 1/2)*sin(f*x + e) - a^2*c*log(-1/2*cos(
f*x + e) + 1/2)*sin(f*x + e) + 2*a^2*c*cos(f*x + e) + 2*(a^2*c*f*x - a^2*c
*cos(f*x + e))*sin(f*x + e))/(f*sin(f*x + e))
```

3.6.6 Sympy [F]

$$\int \csc^2(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx = -a^2c\left(\int\left(-\sin\left(e+fx\right)\csc^2\left(e+fx\right)\right)dx+\int\sin^2\left(e+fx\right)\csc^2\left(e+fx\right)dx+\int\sin^3\left(e+fx\right)\csc^2\left(e+fx\right)dx+\int\left(-\csc^2\left(e+fx\right)\right)dx\right)$$

input
```
integrate(csc(f*x+e)**2*(a+a*sin(f*x+e))**2*(c-c*sin(f*x+e)),x)
```

output
```
-a**2*c*(Integral(-sin(e + f*x)*csc(e + f*x)**2, x) + Integral(sin(e + f*x
)**2*csc(e + f*x)**2, x) + Integral(sin(e + f*x)**3*csc(e + f*x)**2, x) +
Integral(-csc(e + f*x)**2, x))
```

3.6.7 Maxima [A] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 69, normalized size of antiderivative = 1.30

$$\int \csc^2(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx = -\frac{2\,(fx+e)a^2c+a^2c(\log\left(\cos\left(fx+e\right)+1\right)-\log\left(\cos\left(fx+e\right)-1\right))-2\,a^2c\cos\left(fx+e\right)+\frac{2\,a^2c}{\tan(fx+e)}}{2\,f}$$

input
```
integrate(csc(f*x+e)^2*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="m
axima")
```

output
```
-1/2*(2*(f*x + e)*a^2*c + a^2*c*(log(cos(f*x + e) + 1) - log(cos(f*x + e)
- 1)) - 2*a^2*c*cos(f*x + e) + 2*a^2*c/tan(f*x + e))/f
```

3.6.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 129 vs. 2(53) = 106.

Time = 0.32 (sec) , antiderivative size = 129, normalized size of antiderivative = 2.43

$$\int \csc^2(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx = -\frac{6\,(fx+e)a^2c-6\,a^2c\log\left(\left|\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right)\right|\right)-3\,a^2c\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right)+\frac{2\,a^2c\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right)^3+3\,a^2c\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\right.}{\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right)^3+\mathrm{ta}}}{6\,f}$$

input
```
integrate(csc(f*x+e)^2*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="g
iac")
```

output
```
-1/6*(6*(f*x + e)*a^2*c - 6*a^2*c*log(abs(tan(1/2*f*x + 1/2*e))) - 3*a^2*c
*tan(1/2*f*x + 1/2*e) + (2*a^2*c*tan(1/2*f*x + 1/2*e)^3 + 3*a^2*c*tan(1/2*
f*x + 1/2*e)^2 - 10*a^2*c*tan(1/2*f*x + 1/2*e) + 3*a^2*c)/(tan(1/2*f*x + 1
/2*e)^3 + tan(1/2*f*x + 1/2*e)))/f
```

3.6.9 Mupad [B] (verification not implemented)

Time = 12.17 (sec) , antiderivative size = 110, normalized size of antiderivative = 2.08

$$\int \csc^2(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$=\frac{a^2\,c\left(2\,\mathrm{atan}\left(\frac{\sqrt{2}\,\left(\cos\left(\frac{e}{2}+\frac{f\,x}{2}\right)-\sin\left(\frac{e}{2}+\frac{f\,x}{2}\right)\right)}{2\,\cos\left(\frac{e}{2}-\frac{\pi}{4}+\frac{f\,x}{2}\right)}\right)+\ln\left(\frac{\sin\left(\frac{e}{2}+\frac{f\,x}{2}\right)}{\cos\left(\frac{e}{2}+\frac{f\,x}{2}\right)}\right)\right)}{f}$$
$$-\frac{a^2\,c\left(\cos\left(e+f\,x\right)-\frac{\sin(2\,e+2\,f\,x)}{2}\right)}{f\,\sin\left(e+f\,x\right)}$$

input
```
int(((a + a*sin(e + f*x))^2*(c - c*sin(e + f*x)))/sin(e + f*x)^2,x)
```

output
```
(a^2*c*(2*atan((2^(1/2)*(cos(e/2 + (f*x)/2) - sin(e/2 + (f*x)/2)))/(2*cos(
e/2 - pi/4 + (f*x)/2))) + log(sin(e/2 + (f*x)/2)/cos(e/2 + (f*x)/2))))/f -
 (a^2*c*(cos(e + f*x) - sin(2*e + 2*f*x)/2))/(f*sin(e + f*x))
```

3.7 $\int \csc^3(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$

3.7.1 Optimal result

Integrand size = 32, antiderivative size = 64

$$\int \csc^3(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx = -a^2cx+\frac{a^2c\,\text{arctanh}(\cos(e+fx))}{2f}-\frac{a^2c\cot(e+fx)}{f}-\frac{a^2c\cot(e+fx)\csc(e+fx)}{2f}$$

output
```
-a^2*c*x+1/2*a^2*c*arctanh(cos(f*x+e))/f-a^2*c*cot(f*x+e)/f-1/2*a^2*c*cot(
f*x+e)*csc(f*x+e)/f
```

3.7.2 Mathematica [A] (verified)

Time = 0.63 (sec) , antiderivative size = 95, normalized size of antiderivative = 1.48

$$\int \csc^3(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx = -\frac{a^2c\left(8e+8fx+4\cot\left(\frac{1}{2}(e+fx)\right)+\csc^2\left(\frac{1}{2}(e+fx)\right)-4\log\left(\cos\left(\frac{1}{2}(e+fx)\right)\right)+4\log\left(\sin\left(\frac{1}{2}(e+fx)\right)\right.\right.}{8f}$$

input
```
Integrate[Csc[e + f*x]^3*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output
```
-1/8*(a^2*c*(8*e + 8*f*x + 4*Cot[(e + f*x)/2] + Csc[(e + f*x)/2]^2 - 4*Log
[Cos[(e + f*x)/2]] + 4*Log[Sin[(e + f*x)/2]] - Sec[(e + f*x)/2]^2 - 4*Tan[
(e + f*x)/2]))/f
```

3.7.3 Rubi [A] (verified)

Time = 0.31 (sec) , antiderivative size = 64, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.094, Rules used = {3042, 3445, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \csc^3(e+fx)(a\sin(e+fx)+a)^2(c-c\sin(e+fx))\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{(a\sin(e+fx)+a)^2(c-c\sin(e+fx))}{\sin(e+fx)^3}dx$$

$$\Big\downarrow \text{3445}$$

$$\int \left(a^2c\csc^3(e+fx)+a^2c\csc^2(e+fx)-a^2c\csc(e+fx)-a^2c\right)dx$$

$$\Big\downarrow \text{2009}$$

$$\frac{a^2c\,\text{arctanh}(\cos(e+fx))}{2f}-\frac{a^2c\cot(e+fx)}{f}-\frac{a^2c\cot(e+fx)\csc(e+fx)}{2f}+a^2(-c)x$$

input
```
Int[Csc[e + f*x]^3*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output
```
-(a^2*c*x) + (a^2*c*ArcTanh[Cos[e + f*x]])/(2*f) - (a^2*c*Cot[e + f*x])/f
- (a^2*c*Cot[e + f*x]*Csc[e + f*x])/(2*f)
```

3.7.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3445
```
Int[sin[(e_.) + (f_.)*(x_)]^(n_.)*((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m
_.)*((A_.) + (B_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] :> Int[ExpandTrig[si
n[e + f*x]^n*(a + b*sin[e + f*x])^m*(A + B*sin[e + f*x]), x], x] /; FreeQ[{
a, b, e, f, A, B}, x] && EqQ[A*b + a*B, 0] && EqQ[a^2 - b^2, 0] && IntegerQ
[m] && IntegerQ[n]
```

3.7.4 Maple [A] (verified)

Time = 0.78 (sec) , antiderivative size = 73, normalized size of antiderivative = 1.14

method	result
parallelrisch	$\frac{a^{2} c \left(\tan^{2}\left(\frac{f x}{2}+\frac{e}{2}\right)-\left(\cot^{2}\left(\frac{f x}{2}+\frac{e}{2}\right)\right)-8 f x+4 \tan\left(\frac{f x}{2}+\frac{e}{2}\right)-4 \ln\left(\tan\left(\frac{f x}{2}+\frac{e}{2}\right)\right)-4 \cot\left(\frac{f x}{2}+\frac{e}{2}\right)\right)}{8 f}$
derivativedivides	$\frac{-a^{2} c (f x+e)-a^{2} c \ln(\csc(f x+e)-\cot(f x+e))-a^{2} c \cot(f x+e)+a^{2} c \left(-\frac{\csc(f x+e) \cot(f x+e)}{2}+\frac{\ln(\csc(f x+e)-\cot(f x+e))}{2}\right)}{f}$
default	$\frac{-a^{2} c (f x+e)-a^{2} c \ln(\csc(f x+e)-\cot(f x+e))-a^{2} c \cot(f x+e)+a^{2} c \left(-\frac{\csc(f x+e) \cot(f x+e)}{2}+\frac{\ln(\csc(f x+e)-\cot(f x+e))}{2}\right)}{f}$
risch	$-a^{2} c x+\frac{a^{2} c \left(\mathrm{e}^{3 i (f x+e)}+\mathrm{e}^{i (f x+e)}-2 i \,\mathrm{e}^{2 i (f x+e)}+2 i\right)}{f \left(\mathrm{e}^{2 i (f x+e)}-1\right)^{2}}+\frac{a^{2} c \ln\left(\mathrm{e}^{i (f x+e)}+1\right)}{2 f}-\frac{a^{2} c \ln\left(\mathrm{e}^{i (f x+e)}-1\right)}{2 f}$
norman	$\frac{\frac{a^{2} c \left(\tan^{7}\left(\frac{f x}{2}+\frac{e}{2}\right)\right)}{f}-\frac{a^{2} c}{8 f}-\frac{3 a^{2} c \left(\tan^{2}\left(\frac{f x}{2}+\frac{e}{2}\right)\right)}{4 f}-\frac{7 a^{2} c \left(\tan^{6}\left(\frac{f x}{2}+\frac{e}{2}\right)\right)}{8 f}-\frac{11 a^{2} c \left(\tan^{4}\left(\frac{f x}{2}+\frac{e}{2}\right)\right)}{8 f}-\frac{a^{2} c \tan\left(\frac{f x}{2}+\frac{e}{2}\right)}{2 f}-\frac{a^{2} c \left(\tan^{3}\right.}{}}{\tan}$

input
```
int(csc(f*x+e)^3*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x,method=_RETURNVERBO
SE)
```

output
```
1/8*a^2*c*(tan(1/2*f*x+1/2*e)^2-cot(1/2*f*x+1/2*e)^2-8*f*x+4*tan(1/2*f*x+1
/2*e)-4*ln(tan(1/2*f*x+1/2*e))-4*cot(1/2*f*x+1/2*e))/f
```

3.7.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 138 vs. $2(60) = 120$.

Time = 0.28 (sec) , antiderivative size = 138, normalized size of antiderivative = 2.16

$$\int \csc^3(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx =$$
$$-\frac{4\,a^2cfx\cos\left(fx+e\right)^2-4\,a^2cfx-4\,a^2c\cos\left(fx+e\right)\sin\left(fx+e\right)-2\,a^2c\cos\left(fx+e\right)-\left(a^2c\cos\left(fx+\right.\right.}{4\left(f\cos\left(fx+e\right)^2\right.}$$

input
```
integrate(csc(f*x+e)^3*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="f
ricas")
```

output
```
-1/4*(4*a^2*c*f*x*cos(f*x + e)^2 - 4*a^2*c*f*x - 4*a^2*c*cos(f*x + e)*sin(
f*x + e) - 2*a^2*c*cos(f*x + e) - (a^2*c*cos(f*x + e)^2 - a^2*c)*log(1/2*c
os(f*x + e) + 1/2) + (a^2*c*cos(f*x + e)^2 - a^2*c)*log(-1/2*cos(f*x + e)
+ 1/2))/(f*cos(f*x + e)^2 - f)
```

3.7.6 Sympy [F]

$$\int \csc^3(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$= -a^2c\left(\int\left(-\sin\left(e+fx\right)\csc^3\left(e+fx\right)\right)\,dx+\int\sin^2\left(e+fx\right)\csc^3\left(e+fx\right)dx\right.$$
$$\left.+\int\sin^3\left(e+fx\right)\csc^3\left(e+fx\right)dx+\int\left(-\csc^3\left(e+fx\right)\right)\,dx\right)$$

input
```
integrate(csc(f*x+e)**3*(a+a*sin(f*x+e))**2*(c-c*sin(f*x+e)),x)
```

output
```
-a**2*c*(Integral(-sin(e + f*x)*csc(e + f*x)**3, x) + Integral(sin(e + f*x
)**2*csc(e + f*x)**3, x) + Integral(sin(e + f*x)**3*csc(e + f*x)**3, x) +
Integral(-csc(e + f*x)**3, x))
```

3.7.7 Maxima [A] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 105, normalized size of antiderivative = 1.64

$$\int \csc^3(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx =$$
$$-\frac{4\,(fx+e)a^2c - a^2c\Big(\frac{2\,\cos(fx+e)}{\cos(fx+e)^2-1} - \log\left(\cos\left(fx+e\right)+1\right) + \log\left(\cos\left(fx+e\right)-1\right)\Big) - 2\,a^2c(\log\left(\cos\left(fx\right.\right.}{4\,f}$$

input
```
integrate(csc(f*x+e)^3*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="m
axima")
```

output
```
-1/4*(4*(f*x + e)*a^2*c - a^2*c*(2*cos(f*x + e)/(cos(f*x + e)^2 - 1) - log
(cos(f*x + e) + 1) + log(cos(f*x + e) - 1)) - 2*a^2*c*(log(cos(f*x + e) +
1) - log(cos(f*x + e) - 1)) + 4*a^2*c/tan(f*x + e))/f
```

3.7.8 Giac [A] (verification not implemented)

Time = 0.32 (sec) , antiderivative size = 116, normalized size of antiderivative = 1.81

$$\int \csc^3(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$= \frac{a^2c\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right)^2 - 8\,(fx+e)a^2c - 4\,a^2c\log\left(\left|\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right)\right|\right) + 4\,a^2c\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right) + \frac{6\,a^2c\tan(\frac{1}{2}}{}}{8\,f}$$

input
```
integrate(csc(f*x+e)^3*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="g
iac")
```

output
```
1/8*(a^2*c*tan(1/2*f*x + 1/2*e)^2 - 8*(f*x + e)*a^2*c - 4*a^2*c*log(abs(ta
n(1/2*f*x + 1/2*e))) + 4*a^2*c*tan(1/2*f*x + 1/2*e) + (6*a^2*c*tan(1/2*f*x
 + 1/2*e)^2 - 4*a^2*c*tan(1/2*f*x + 1/2*e) - a^2*c)/tan(1/2*f*x + 1/2*e)^2
)/f
```

3.7.9 Mupad [B] (verification not implemented)

Time = 12.17 (sec) , antiderivative size = 163, normalized size of antiderivative = 2.55

$$\int \csc^3(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$=\frac{a^2\,c\tan\left(\frac{e}{2}+\frac{f\,x}{2}\right)}{2\,f}-\frac{a^2\,c\,\ln\left(\frac{\sin\left(\frac{e}{2}+\frac{f\,x}{2}\right)}{\cos\left(\frac{e}{2}+\frac{f\,x}{2}\right)}\right)}{2\,f}-\frac{2\,a^2\,c\,\mathrm{atan}\left(\frac{2\,\cos\left(\frac{e}{2}+\frac{f\,x}{2}\right)+\sin\left(\frac{e}{2}+\frac{f\,x}{2}\right)}{\cos\left(\frac{e}{2}+\frac{f\,x}{2}\right)-2\,\sin\left(\frac{e}{2}+\frac{f\,x}{2}\right)}\right)}{f}$$
$$-\frac{a^2\,c\cot\left(\frac{e}{2}+\frac{f\,x}{2}\right)}{2\,f}-\frac{a^2\,c\cot\left(\frac{e}{2}+\frac{f\,x}{2}\right)^2}{8\,f}+\frac{a^2\,c\tan\left(\frac{e}{2}+\frac{f\,x}{2}\right)^2}{8\,f}$$

input
```
int(((a + a*sin(e + f*x))^2*(c - c*sin(e + f*x)))/sin(e + f*x)^3,x)
```

output
```
(a^2*c*tan(e/2 + (f*x)/2))/(2*f) - (a^2*c*log(sin(e/2 + (f*x)/2)/cos(e/2 +
 (f*x)/2)))/(2*f) - (2*a^2*c*atan((2*cos(e/2 + (f*x)/2) + sin(e/2 + (f*x)/
2))/(cos(e/2 + (f*x)/2) - 2*sin(e/2 + (f*x)/2))))/f - (a^2*c*cot(e/2 + (f*
x)/2))/(2*f) - (a^2*c*cot(e/2 + (f*x)/2)^2)/(8*f) + (a^2*c*tan(e/2 + (f*x)
/2)^2)/(8*f)
```

3.8 $\int \csc^4(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$

3.8.1 Optimal result

Integrand size = 32, antiderivative size = 61

$$\int \csc^4(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx = \frac{a^2c\,\text{arctanh}(\cos(e+fx))}{2f} - \frac{a^2c\cot^3(e+fx)}{3f} - \frac{a^2c\cot(e+fx)\csc(e+fx)}{2f}$$

output
```
1/2*a^2*c*arctanh(cos(f*x+e))/f-1/3*a^2*c*cot(f*x+e)^3/f-1/2*a^2*c*cot(f*x
+e)*csc(f*x+e)/f
```

3.8.2 Mathematica [B] (verified)

Leaf count is larger than twice the leaf count of optimal. 172 vs. 2(61) = 122.

Time = 0.20 (sec) , antiderivative size = 172, normalized size of antiderivative = 2.82

$$\int \csc^4(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx = a^2c\left(\frac{\cot\left(\frac{1}{2}(e+fx)\right)}{6f} - \frac{\csc^2\left(\frac{1}{2}(e+fx)\right)}{8f} - \frac{\cot\left(\frac{1}{2}(e+fx)\right)\csc^2\left(\frac{1}{2}(e+fx)\right)}{24f} + \frac{\log\left(\cos\left(\frac{1}{2}(e+fx)\right)\right)}{2f} - \frac{\log\left(\sin\left(\frac{1}{2}(e+fx)\right)\right)}{2f} + \frac{\sec^2\left(\frac{1}{2}(e+fx)\right)}{8f} - \frac{\tan\left(\frac{1}{2}(e+fx)\right)}{6f} + \frac{\sec^2\left(\frac{1}{2}(e+fx)\right)\tan\left(\frac{1}{2}(e+fx)\right)}{24f}\right)$$

input
```
Integrate[Csc[e + f*x]^4*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output
```
a^2*c*(Cot[(e + f*x)/2]/(6*f) - Csc[(e + f*x)/2]^2/(8*f) - (Cot[(e + f*x)/
2]*Csc[(e + f*x)/2]^2)/(24*f) + Log[Cos[(e + f*x)/2]]/(2*f) - Log[Sin[(e +
 f*x)/2]]/(2*f) + Sec[(e + f*x)/2]^2/(8*f) - Tan[(e + f*x)/2]/(6*f) + (Sec
[(e + f*x)/2]^2*Tan[(e + f*x)/2])/(24*f))
```

3.8.3 Rubi [A] (verified)

Time = 0.54 (sec) , antiderivative size = 64, normalized size of antiderivative = 1.05, number of steps used = 11, number of rules used = 10, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.312, Rules used = {3042, 3429, 3042, 3185, 3042, 3087, 15, 3091, 3042, 4257}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \csc^4(e+fx)(a\sin(e+fx)+a)^2(c-c\sin(e+fx))\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{(a\sin(e+fx)+a)^2(c-c\sin(e+fx))}{\sin(e+fx)^4}dx$$

$$\Big\downarrow \text{3429}$$

$$a^2c^2\int \frac{\cot^4(e+fx)}{c-c\sin(e+fx)}dx$$

$$\Big\downarrow \text{3042}$$

$$a^2c^2\int \frac{1}{(c-c\sin(e+fx))\tan(e+fx)^4}dx$$

$$\Big\downarrow \text{3185}$$

$$a^2c^2\left(\frac{\int \cot^2(e+fx)\csc^2(e+fx)dx}{c}+\frac{\int \cot^2(e+fx)\csc(e+fx)dx}{c}\right)$$

$$\Big\downarrow \text{3042}$$

$$a^2c^2\left(\frac{\int \sec\left(e+fx-\frac{\pi}{2}\right)\tan\left(e+fx-\frac{\pi}{2}\right)^2dx}{c}+\frac{\int \sec\left(e+fx-\frac{\pi}{2}\right)^2\tan\left(e+fx-\frac{\pi}{2}\right)^2dx}{c}\right)$$

$$\Big\downarrow \text{3087}$$

$$a^2c^2\left(\frac{\int \cot^2(e+fx)d(-\cot(e+fx))}{cf}+\frac{\int \sec\left(e+fx-\frac{\pi}{2}\right)\tan\left(e+fx-\frac{\pi}{2}\right)^2dx}{c}\right)$$

$$\Big\downarrow \textbf{15}$$

$$a^2c^2\left(\frac{\int \sec\left(e+fx-\frac{\pi}{2}\right)\tan\left(e+fx-\frac{\pi}{2}\right)^2dx}{c}-\frac{\cot^3(e+fx)}{3cf}\right)$$

$$\Big\downarrow \textbf{3091}$$

$$a^2c^2\left(\frac{-\frac{1}{2}\int \csc(e+fx)dx-\frac{\cot(e+fx)\csc(e+fx)}{2f}}{c}-\frac{\cot^3(e+fx)}{3cf}\right)$$

$$\Big\downarrow \textbf{3042}$$

$$a^2c^2\left(\frac{-\frac{1}{2}\int \csc(e+fx)dx-\frac{\cot(e+fx)\csc(e+fx)}{2f}}{c}-\frac{\cot^3(e+fx)}{3cf}\right)$$

$$\Big\downarrow \textbf{4257}$$

$$a^2c^2\left(\frac{\frac{\text{arctanh}(\cos(e+fx))}{2f}-\frac{\cot(e+fx)\csc(e+fx)}{2f}}{c}-\frac{\cot^3(e+fx)}{3cf}\right)$$

input
```
Int[Csc[e + f*x]^4*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output
```
a^2*c^2*(-1/3*Cot[e + f*x]^3/(c*f) + (ArcTanh[Cos[e + f*x]]/(2*f) - (Cot[e
 + f*x]*Csc[e + f*x])/(2*f))/c)
```

3.8.3.1 Defintions of rubi rules used

rule 15
```
Int[(a_.)*(x_)^(m_.), x_Symbol] :> Simp[a*(x^(m + 1)/(m + 1)), x] /; FreeQ[
{a, m}, x] && NeQ[m, -1]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3087
```
Int[sec[(e_.) + (f_.)*(x_)]^(m_)*((b_.)*tan[(e_.) + (f_.)*(x_)])^(n_.), x_S
ymbol] :> Simp[1/f   Subst[Int[(b*x)^n*(1 + x^2)^(m/2 - 1), x], x, Tan[e +
f*x]], x] /; FreeQ[{b, e, f, n}, x] && IntegerQ[m/2] &&  !(IntegerQ[(n - 1)
/2] && LtQ[0, n, m - 1])
```

rule 3091
```
Int[((a_.)*sec[(e_.) + (f_.)*(x_)])^(m_.)*((b_.)*tan[(e_.) + (f_.)*(x_)])^(
n_), x_Symbol] :> Simp[b*(a*Sec[e + f*x])^m*((b*Tan[e + f*x])^(n - 1)/(f*(m
 + n - 1))), x] - Simp[b^2*((n - 1)/(m + n - 1))   Int[(a*Sec[e + f*x])^m*(
b*Tan[e + f*x])^(n - 2), x], x] /; FreeQ[{a, b, e, f, m}, x] && GtQ[n, 1] &
& NeQ[m + n - 1, 0] && IntegersQ[2*m, 2*n]
```

rule 3185
```
Int[((g_.)*tan[(e_.) + (f_.)*(x_)])^(p_.)/((a_) + (b_.)*sin[(e_.) + (f_.)*(
x_)]), x_Symbol] :> Simp[1/a   Int[Sec[e + f*x]^2*(g*Tan[e + f*x])^p, x], x
] - Simp[1/(b*g)   Int[Sec[e + f*x]*(g*Tan[e + f*x])^(p + 1), x], x] /; Fre
eQ[{a, b, e, f, g, p}, x] && EqQ[a^2 - b^2, 0] && NeQ[p, -1]
```

rule 3429
```
Int[sin[(e_.) + (f_.)*(x_)]^(p_)*((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_
.)*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])^(n_.), x_Symbol] :> Simp[a^n*c^n
  Int[Tan[e + f*x]^p*(a + b*Sin[e + f*x])^(m - n), x], x] /; FreeQ[{a, b, c
, d, e, f, m}, x] && EqQ[b*c + a*d, 0] && EqQ[a^2 - b^2, 0] && EqQ[p + 2*n,
 0] && IntegerQ[n]
```

rule 4257
```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> Simp[-ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

3.8.4 Maple [A] (verified)

Time = 0.92 (sec) , antiderivative size = 95, normalized size of antiderivative = 1.56

method	result
parallelrisch	$\frac{a^2 c\left(\tan^3\left(\frac{fx}{2}+\frac{e}{2}\right)-\left(\cot^3\left(\frac{fx}{2}+\frac{e}{2}\right)\right)+3\left(\tan^2\left(\frac{fx}{2}+\frac{e}{2}\right)\right)-3\left(\cot^2\left(\frac{fx}{2}+\frac{e}{2}\right)\right)-3\tan\left(\frac{fx}{2}+\frac{e}{2}\right)-12\ln\left(\tan\left(\frac{fx}{2}+\frac{e}{2}\right)\right)+3\text{c}}{24f}$
derivativedivides	$\frac{-a^2c\ln(\csc(fx+e)-\cot(fx+e))+a^2c\cot(fx+e)+a^2c\left(-\frac{\csc(fx+e)\cot(fx+e)}{2}+\frac{\ln(\csc(fx+e)-\cot(fx+e))}{2}\right)+a^2c\left(-\frac{2}{3}-\frac{(\text{cs}}{}\right.}{f}$
default	$\frac{-a^2c\ln(\csc(fx+e)-\cot(fx+e))+a^2c\cot(fx+e)+a^2c\left(-\frac{\csc(fx+e)\cot(fx+e)}{2}+\frac{\ln(\csc(fx+e)-\cot(fx+e))}{2}\right)+a^2c\left(-\frac{2}{3}-\frac{(\text{cs}}{}\right.}{f}$
risch	$\frac{a^2c\left(6i e^{4i(fx+e)}+3e^{5i(fx+e)}+2i-3e^{i(fx+e)}\right)}{3f\left(e^{2i(fx+e)}-1\right)^3}+\frac{a^2c\ln\left(e^{i(fx+e)}+1\right)}{2f}-\frac{a^2c\ln\left(e^{i(fx+e)}-1\right)}{2f}$
norman	$\frac{-\frac{a^2c}{24f}-\frac{3a^2c\left(\tan^3\left(\frac{fx}{2}+\frac{e}{2}\right)\right)}{4f}-\frac{7a^2c\left(\tan^7\left(\frac{fx}{2}+\frac{e}{2}\right)\right)}{8f}-\frac{11a^2c\left(\tan^5\left(\frac{fx}{2}+\frac{e}{2}\right)\right)}{8f}-\frac{a^2c\tan\left(\frac{fx}{2}+\frac{e}{2}\right)}{8f}+\frac{a^2c\left(\tan^4\left(\frac{fx}{2}+\frac{e}{2}\right)\right)}{8f}-\frac{a^2c\left(\tan\right.}{}}{\tan\left(\frac{fx}{2}+\frac{e}{2}\right)^3\left(1+\tan^2\left(\frac{fx}{2}+\frac{e}{2}\right)\right)^3}$

input
```
int(csc(f*x+e)^4*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x,method=_RETURNVERBO
SE)
```

output
```
1/24*a^2*c*(tan(1/2*f*x+1/2*e)^3-cot(1/2*f*x+1/2*e)^3+3*tan(1/2*f*x+1/2*e)
^2-3*cot(1/2*f*x+1/2*e)^2-3*tan(1/2*f*x+1/2*e)-12*ln(tan(1/2*f*x+1/2*e))+3
*cot(1/2*f*x+1/2*e))/f
```

3.8.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 137 vs. $2(55) = 110$.

Time = 0.28 (sec) , antiderivative size = 137, normalized size of antiderivative = 2.25

$$\int \csc^4(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$= \frac{4\,a^2c\cos(fx+e)^3+6\,a^2c\cos(fx+e)\sin(fx+e)+3\left(a^2c\cos(fx+e)^2-a^2c\right)\log\left(\frac{1}{2}\cos(fx+e)+\frac{1}{2}\right)}{12\left(f\cos(fx+e)^2-f\right)\sin(f}$$

input
```
integrate(csc(f*x+e)^4*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="f
ricas")
```

output
```
1/12*(4*a^2*c*cos(f*x + e)^3 + 6*a^2*c*cos(f*x + e)*sin(f*x + e) + 3*(a^2*
c*cos(f*x + e)^2 - a^2*c)*log(1/2*cos(f*x + e) + 1/2)*sin(f*x + e) - 3*(a^
2*c*cos(f*x + e)^2 - a^2*c)*log(-1/2*cos(f*x + e) + 1/2)*sin(f*x + e))/((f
*cos(f*x + e)^2 - f)*sin(f*x + e))
```

3.8.6 Sympy [F]

$$\int \csc^4(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$= -a^2c\left(\int\left(-\sin\left(e+fx\right)\csc^4\left(e+fx\right)\right)\,dx+\int\sin^2\left(e+fx\right)\csc^4\left(e+fx\right)dx+\int\sin^3\left(e+fx\right)\csc^4\left(e+fx\right)dx+\int\left(-\csc^4\left(e+fx\right)\right)\,dx\right)$$

input
```
integrate(csc(f*x+e)**4*(a+a*sin(f*x+e))**2*(c-c*sin(f*x+e)),x)
```

output
```
-a**2*c*(Integral(-sin(e + f*x)*csc(e + f*x)**4, x) + Integral(sin(e + f*x
)**2*csc(e + f*x)**4, x) + Integral(sin(e + f*x)**3*csc(e + f*x)**4, x) +
Integral(-csc(e + f*x)**4, x))
```

3.8.7 Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 120 vs. $2(55) = 110$.

Time = 0.20 (sec) , antiderivative size = 120, normalized size of antiderivative = 1.97

$$\int \csc^4(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$= \frac{3\,a^2c\left(\frac{2\,\cos(fx+e)}{\cos(fx+e)^2-1}-\log\left(\cos\left(fx+e\right)+1\right)+\log\left(\cos\left(fx+e\right)-1\right)\right)+6\,a^2c(\log\left(\cos\left(fx+e\right)+1\right)-\log\left(\right.}{12\,f}$$

input
```
integrate(csc(f*x+e)^4*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="m
axima")
```

output
```
1/12*(3*a^2*c*(2*cos(f*x + e)/(cos(f*x + e)^2 - 1) - log(cos(f*x + e) + 1)
 + log(cos(f*x + e) - 1)) + 6*a^2*c*(log(cos(f*x + e) + 1) - log(cos(f*x +
 e) - 1)) + 12*a^2*c/tan(f*x + e) - 4*(3*tan(f*x + e)^2 + 1)*a^2*c/tan(f*x
 + e)^3)/f
```

3.8.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 139 vs. $2(55) = 110$.

Time = 0.31 (sec) , antiderivative size = 139, normalized size of antiderivative = 2.28

$$\int \csc^4(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx = \frac{a^2c\tan\left(\frac{1}{2}fx+\frac{1}{2}e\right)^3+3a^2c\tan\left(\frac{1}{2}fx+\frac{1}{2}e\right)^2-12a^2c\log\left(\left|\tan\left(\frac{1}{2}fx+\frac{1}{2}e\right)\right|\right)-3a^2c\tan\left(\frac{1}{2}fx+\frac{1}{2}e\right)+}{24f}$$

input
```
integrate(csc(f*x+e)^4*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="g
iac")
```

output
```
1/24*(a^2*c*tan(1/2*f*x + 1/2*e)^3 + 3*a^2*c*tan(1/2*f*x + 1/2*e)^2 - 12*a
^2*c*log(abs(tan(1/2*f*x + 1/2*e))) - 3*a^2*c*tan(1/2*f*x + 1/2*e) + (22*a
^2*c*tan(1/2*f*x + 1/2*e)^3 + 3*a^2*c*tan(1/2*f*x + 1/2*e)^2 - 3*a^2*c*tan
(1/2*f*x + 1/2*e) - a^2*c)/tan(1/2*f*x + 1/2*e)^3)/f
```

3.8.9 Mupad [B] (verification not implemented)

Time = 11.69 (sec) , antiderivative size = 132, normalized size of antiderivative = 2.16

$$\begin{aligned}\int \csc^4(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx &= \frac{a^2\,c\tan\left(\frac{e}{2}+\frac{f\,x}{2}\right)^2}{8\,f}-\frac{a^2\,c\tan\left(\frac{e}{2}+\frac{f\,x}{2}\right)}{8\,f}\\&\quad-\frac{\cot\left(\frac{e}{2}+\frac{f\,x}{2}\right)^3\left(-c\,a^2\tan\left(\frac{e}{2}+\frac{f\,x}{2}\right)^2+c\,a^2\tan\left(\frac{e}{2}+\frac{f\,x}{2}\right)+\frac{c\,a^2}{3}\right)}{8\,f}\\&\quad+\frac{a^2\,c\tan\left(\frac{e}{2}+\frac{f\,x}{2}\right)^3}{24\,f}-\frac{a^2\,c\,\ln\left(\tan\left(\frac{e}{2}+\frac{f\,x}{2}\right)\right)}{2\,f}\end{aligned}$$

input
```
int(((a + a*sin(e + f*x))^2*(c - c*sin(e + f*x)))/sin(e + f*x)^4,x)
```

output
```
(a^2*c*tan(e/2 + (f*x)/2)^2)/(8*f) - (a^2*c*tan(e/2 + (f*x)/2))/(8*f) - (c
ot(e/2 + (f*x)/2)^3*((a^2*c)/3 + a^2*c*tan(e/2 + (f*x)/2) - a^2*c*tan(e/2
+ (f*x)/2)^2))/(8*f) + (a^2*c*tan(e/2 + (f*x)/2)^3)/(24*f) - (a^2*c*log(ta
n(e/2 + (f*x)/2)))/(2*f)
```

3.9 $\int \csc^5(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$

3.9.1 Optimal result

Integrand size = 32, antiderivative size = 86

$$\int \csc^5(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx = \frac{a^2c\,\text{arctanh}(\cos(e+fx))}{8f} - \frac{a^2c\cot^3(e+fx)}{3f} + \frac{a^2c\cot(e+fx)\csc(e+fx)}{8f} - \frac{a^2c\cot(e+fx)\csc^3(e+fx)}{4f}$$

output
```
1/8*a^2*c*arctanh(cos(f*x+e))/f-1/3*a^2*c*cot(f*x+e)^3/f+1/8*a^2*c*cot(f*x
+e)*csc(f*x+e)/f-1/4*a^2*c*cot(f*x+e)*csc(f*x+e)^3/f
```

3.9.2 Mathematica [B] (verified)

Leaf count is larger than twice the leaf count of optimal. 179 vs. 2(86) = 172.

Time = 0.22 (sec) , antiderivative size = 179, normalized size of antiderivative = 2.08

$$\int \csc^5(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx = \frac{a^2c\cot(e+fx)}{3f}+\frac{a^2c\csc^2\left(\frac{1}{2}(e+fx)\right)}{32f}-\frac{a^2c\csc^4\left(\frac{1}{2}(e+fx)\right)}{64f}-\frac{a^2c\cot(e+fx)\csc^2(e+fx)}{3f}+\frac{a^2c\log\left(\cos\left(\frac{1}{2}(e+fx)\right)\right)}{8f}-\frac{a^2c\log\left(\sin\left(\frac{1}{2}(e+fx)\right)\right)}{8f}-\frac{a^2c\sec^2\left(\frac{1}{2}(e+fx)\right)}{32f}+\frac{a^2c\sec^4\left(\frac{1}{2}(e+fx)\right)}{64f}$$

input
```
Integrate[Csc[e + f*x]^5*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output
```
(a^2*c*Cot[e + f*x])/(3*f) + (a^2*c*Csc[(e + f*x)/2]^2)/(32*f) - (a^2*c*Cs
c[(e + f*x)/2]^4)/(64*f) - (a^2*c*Cot[e + f*x]*Csc[e + f*x]^2)/(3*f) + (a^
2*c*Log[Cos[(e + f*x)/2]])/(8*f) - (a^2*c*Log[Sin[(e + f*x)/2]])/(8*f) - (
a^2*c*Sec[(e + f*x)/2]^2)/(32*f) + (a^2*c*Sec[(e + f*x)/2]^4)/(64*f)
```

3.9.3 Rubi [A] (verified)

Time = 0.35 (sec) , antiderivative size = 86, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.094, Rules used = {3042, 3445, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \csc^5(e+fx)(a\sin(e+fx)+a)^2(c-c\sin(e+fx))\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{(a\sin(e+fx)+a)^2(c-c\sin(e+fx))}{\sin(e+fx)^5}dx$$

$$\Big\downarrow \text{3445}$$

$$\int \left(a^2c\csc^5(e+fx)+a^2c\csc^4(e+fx)-a^2c\csc^3(e+fx)-a^2c\csc^2(e+fx)\right)dx$$

$$\Big\downarrow \text{2009}$$

$$\frac{a^2 c\operatorname{arctanh}(\cos(e+fx))}{8f}-\frac{a^2 c\cot^3(e+fx)}{3f}-\frac{a^2 c\cot(e+fx)\csc^3(e+fx)}{4f}+\frac{a^2 c\cot(e+fx)\csc(e+fx)}{8f}$$

input
```
Int[Csc[e + f*x]^5*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output
```
(a^2*c*ArcTanh[Cos[e + f*x]])/(8*f) - (a^2*c*Cot[e + f*x]^3)/(3*f) + (a^2*
c*Cot[e + f*x]*Csc[e + f*x])/(8*f) - (a^2*c*Cot[e + f*x]*Csc[e + f*x]^3)/(
4*f)
```

3.9.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3445
```
Int[sin[(e_.) + (f_.)*(x_)]^(n_.)*((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m
_.)*((A_.) + (B_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] :> Int[ExpandTrig[si
n[e + f*x]^n*(a + b*sin[e + f*x])^m*(A + B*sin[e + f*x]), x], x] /; FreeQ[{
a, b, e, f, A, B}, x] && EqQ[A*b + a*B, 0] && EqQ[a^2 - b^2, 0] && IntegerQ
[m] && IntegerQ[n]
```

3.9.4 Maple [A] (verified)

Time = 1.07 (sec) , antiderivative size = 109, normalized size of antiderivative = 1.27

method	result
parallelrisch	$-\frac{a^2c\left(-3\left(\tan^8\left(\frac{fx}{2}+\frac{e}{2}\right)\right)-8\left(\tan^7\left(\frac{fx}{2}+\frac{e}{2}\right)\right)+24\ln\left(\tan\left(\frac{fx}{2}+\frac{e}{2}\right)\right)\left(\tan^4\left(\frac{fx}{2}+\frac{e}{2}\right)\right)+24\left(\tan^5\left(\frac{fx}{2}+\frac{e}{2}\right)\right)-24\left(\tan^3\left(\right.\right.\right.}{192f\tan\left(\frac{fx}{2}+\frac{e}{2}\right)^4}$
derivativedivides	$\frac{a^2c\cot(fx+e)-a^2c\left(-\frac{\csc(fx+e)\cot(fx+e)}{2}+\frac{\ln(\csc(fx+e)-\cot(fx+e))}{2}\right)+a^2c\left(-\frac{2}{3}-\frac{\left(\csc^2(fx+e)\right)}{3}\right)\cot(fx+e)+a^2c\left(\left(\left(-\right.\right.\right.}{f}$
default	$\frac{a^2c\cot(fx+e)-a^2c\left(-\frac{\csc(fx+e)\cot(fx+e)}{2}+\frac{\ln(\csc(fx+e)-\cot(fx+e))}{2}\right)+a^2c\left(-\frac{2}{3}-\frac{\left(\csc^2(fx+e)\right)}{3}\right)\cot(fx+e)+a^2c\left(\left(\left(-\right.\right.\right.}{f}$
risch	$-\frac{a^2c\left(3\,e^{7i(fx+e)}+21\,e^{5i(fx+e)}-24ie^{6i(fx+e)}+21\,e^{3i(fx+e)}+24ie^{4i(fx+e)}+3\,e^{i(fx+e)}-8ie^{2i(fx+e)}+8i\right)}{12f\left(e^{2i(fx+e)}-1\right)^4}+\frac{a^2c\ln\left(e^{i(\right.}}{8}$
norman	$\frac{-\frac{a^2c}{64f}-\frac{a^2c\left(\tan^8\left(\frac{fx}{2}+\frac{e}{2}\right)\right)}{8f}-\frac{3a^2c\left(\tan^4\left(\frac{fx}{2}+\frac{e}{2}\right)\right)}{32f}-\frac{5a^2c\left(\tan^6\left(\frac{fx}{2}+\frac{e}{2}\right)\right)}{32f}-\frac{a^2c\tan\left(\frac{fx}{2}+\frac{e}{2}\right)}{24f}-\frac{3a^2c\left(\tan^2\left(\frac{fx}{2}+\frac{e}{2}\right)\right)}{64f}+\frac{a^2c\left(\tan^5\right.}{}}{\tan\left(\frac{fx}{2}+\frac{e}{2}\right)^4\left(1+\tan^2\left(\frac{fx}{2}+\right.\right.}$

input
```
int(csc(f*x+e)^5*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x,method=_RETURNVERBO
SE)
```

output
```
-1/192*a^2*c*(-3*tan(1/2*f*x+1/2*e)^8-8*tan(1/2*f*x+1/2*e)^7+24*ln(tan(1/2
*f*x+1/2*e))*tan(1/2*f*x+1/2*e)^4+24*tan(1/2*f*x+1/2*e)^5-24*tan(1/2*f*x+1
/2*e)^3+8*tan(1/2*f*x+1/2*e)+3)/f/tan(1/2*f*x+1/2*e)^4
```

3.9.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 166 vs. 2(78) = 156.

Time = 0.26 (sec) , antiderivative size = 166, normalized size of antiderivative = 1.93

$$\int \csc^5(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx =$$
$$-\frac{16\,a^2c\cos\left(fx+e\right)^3\sin\left(fx+e\right)+6\,a^2c\cos\left(fx+e\right)^3+6\,a^2c\cos\left(fx+e\right)-3\,\left(a^2c\cos\left(fx+e\right)^4-2\,a^2\right.}{48\,\left(f\cos\left(f\right.\right.}$$

input
```
integrate(csc(f*x+e)^5*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="f
ricas")
```

output
```
-1/48*(16*a^2*c*cos(f*x + e)^3*sin(f*x + e) + 6*a^2*c*cos(f*x + e)^3 + 6*a
^2*c*cos(f*x + e) - 3*(a^2*c*cos(f*x + e)^4 - 2*a^2*c*cos(f*x + e)^2 + a^2
*c)*log(1/2*cos(f*x + e) + 1/2) + 3*(a^2*c*cos(f*x + e)^4 - 2*a^2*c*cos(f*
x + e)^2 + a^2*c)*log(-1/2*cos(f*x + e) + 1/2))/(f*cos(f*x + e)^4 - 2*f*co
s(f*x + e)^2 + f)
```

3.9.6 Sympy [F(-1)]

Timed out.

$$\int \csc^5(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx = \text{Timed out}$$

input
```
integrate(csc(f*x+e)**5*(a+a*sin(f*x+e))**2*(c-c*sin(f*x+e)),x)
```

output
```
Timed out
```

3.9.7 Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 165 vs. 2(78) = 156.

Time = 0.20 (sec) , antiderivative size = 165, normalized size of antiderivative = 1.92

$$\int \csc^5(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$= \frac{3\,a^2c\left(\frac{2\left(3\,\cos(fx+e)^3-5\,\cos(fx+e)\right)}{\cos(fx+e)^4-2\,\cos(fx+e)^2+1}-3\,\log\left(\cos\left(fx+e\right)+1\right)+3\,\log\left(\cos\left(fx+e\right)-1\right)\right)-12\,a^2c\left(\frac{2\,\cos(fx+e}{\cos(fx+e)^2-}\right.}{48\,f}$$

input
```
integrate(csc(f*x+e)^5*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="m
axima")
```

output
```
1/48*(3*a^2*c*(2*(3*cos(f*x + e)^3 - 5*cos(f*x + e))/(cos(f*x + e)^4 - 2*c
os(f*x + e)^2 + 1) - 3*log(cos(f*x + e) + 1) + 3*log(cos(f*x + e) - 1)) -
12*a^2*c*(2*cos(f*x + e)/(cos(f*x + e)^2 - 1) - log(cos(f*x + e) + 1) + lo
g(cos(f*x + e) - 1)) + 48*a^2*c/tan(f*x + e) - 16*(3*tan(f*x + e)^2 + 1)*a
^2*c/tan(f*x + e)^3)/f
```

3.9.8 Giac [A] (verification not implemented)

Time = 0.40 (sec) , antiderivative size = 140, normalized size of antiderivative = 1.63

$$\int \csc^5(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$=\frac{3\,a^2c\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right)^4+8\,a^2c\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right)^3-24\,a^2c\log\left(\left|\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right)\right|\right)-24\,a^2c\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right.}{192\,f}$$

input
```
integrate(csc(f*x+e)^5*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="g
iac")
```

output
```
1/192*(3*a^2*c*tan(1/2*f*x + 1/2*e)^4 + 8*a^2*c*tan(1/2*f*x + 1/2*e)^3 - 2
4*a^2*c*log(abs(tan(1/2*f*x + 1/2*e))) - 24*a^2*c*tan(1/2*f*x + 1/2*e) + (
50*a^2*c*tan(1/2*f*x + 1/2*e)^4 + 24*a^2*c*tan(1/2*f*x + 1/2*e)^3 - 8*a^2*
c*tan(1/2*f*x + 1/2*e) - 3*a^2*c)/tan(1/2*f*x + 1/2*e)^4)/f
```

3.9.9 Mupad [B] (verification not implemented)

Time = 11.53 (sec) , antiderivative size = 133, normalized size of antiderivative = 1.55

$$\int \csc^5(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$=\frac{a^2\,c\tan\left(\frac{e}{2}+\frac{f\,x}{2}\right)^3}{24\,f}-\frac{a^2\,c\tan\left(\frac{e}{2}+\frac{f\,x}{2}\right)}{8\,f}$$
$$-\frac{\cot\left(\frac{e}{2}+\frac{f\,x}{2}\right)^4\left(-2\,c\,a^2\tan\left(\frac{e}{2}+\frac{f\,x}{2}\right)^3+\frac{2\,c\,a^2\tan\left(\frac{e}{2}+\frac{f\,x}{2}\right)}{3}+\frac{c\,a^2}{4}\right)}{16\,f}$$
$$+\frac{a^2\,c\tan\left(\frac{e}{2}+\frac{f\,x}{2}\right)^4}{64\,f}-\frac{a^2\,c\,\ln\left(\tan\left(\frac{e}{2}+\frac{f\,x}{2}\right)\right)}{8\,f}$$

input
```
int(((a + a*sin(e + f*x))^2*(c - c*sin(e + f*x)))/sin(e + f*x)^5,x)
```

output
```
(a^2*c*tan(e/2 + (f*x)/2)^3)/(24*f) - (a^2*c*tan(e/2 + (f*x)/2))/(8*f) - (
cot(e/2 + (f*x)/2)^4*((a^2*c)/4 + (2*a^2*c*tan(e/2 + (f*x)/2))/3 - 2*a^2*c
*tan(e/2 + (f*x)/2)^3))/(16*f) + (a^2*c*tan(e/2 + (f*x)/2)^4)/(64*f) - (a^
2*c*log(tan(e/2 + (f*x)/2)))/(8*f)
```

3.10 $\int \csc^6(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$

3.10.1 Optimal result

Integrand size = 32, antiderivative size = 105

$$\begin{aligned}\int \csc^6(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx &= \frac{a^2c\,\text{arctanh}(\cos(e+fx))}{8f}-\frac{a^2c\cot^3(e+fx)}{3f}-\frac{a^2c\cot^5(e+fx)}{5f}\\&\quad+\frac{a^2c\cot(e+fx)\csc(e+fx)}{8f}-\frac{a^2c\cot(e+fx)\csc^3(e+fx)}{4f}\end{aligned}$$

output
```
1/8*a^2*c*arctanh(cos(f*x+e))/f-1/3*a^2*c*cot(f*x+e)^3/f-1/5*a^2*c*cot(f*x
+e)^5/f+1/8*a^2*c*cot(f*x+e)*csc(f*x+e)/f-1/4*a^2*c*cot(f*x+e)*csc(f*x+e)^
3/f
```

3.10.2 Mathematica [A] (verified)

Time = 0.11 (sec) , antiderivative size = 204, normalized size of antiderivative = 1.94

$$\int \csc^6(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$= \frac{2a^2c\cot(e+fx)}{15f}+\frac{a^2c\csc^2\left(\frac{1}{2}(e+fx)\right)}{32f}-\frac{a^2c\csc^4\left(\frac{1}{2}(e+fx)\right)}{64f}$$
$$+\frac{a^2c\cot(e+fx)\csc^2(e+fx)}{15f}-\frac{a^2c\cot(e+fx)\csc^4(e+fx)}{5f}$$
$$+\frac{a^2c\log\left(\cos\left(\frac{1}{2}(e+fx)\right)\right)}{8f}-\frac{a^2c\log\left(\sin\left(\frac{1}{2}(e+fx)\right)\right)}{8f}$$
$$-\frac{a^2c\sec^2\left(\frac{1}{2}(e+fx)\right)}{32f}+\frac{a^2c\sec^4\left(\frac{1}{2}(e+fx)\right)}{64f}$$

input
```
Integrate[Csc[e + f*x]^6*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output
```
(2*a^2*c*Cot[e + f*x])/(15*f) + (a^2*c*Csc[(e + f*x)/2]^2)/(32*f) - (a^2*c
*Csc[(e + f*x)/2]^4)/(64*f) + (a^2*c*Cot[e + f*x]*Csc[e + f*x]^2)/(15*f) -
 (a^2*c*Cot[e + f*x]*Csc[e + f*x]^4)/(5*f) + (a^2*c*Log[Cos[(e + f*x)/2]])
/(8*f) - (a^2*c*Log[Sin[(e + f*x)/2]])/(8*f) - (a^2*c*Sec[(e + f*x)/2]^2)/
(32*f) + (a^2*c*Sec[(e + f*x)/2]^4)/(64*f)
```

3.10.3 Rubi [A] (verified)

Time = 0.37 (sec) , antiderivative size = 105, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.094, Rules used = {3042, 3445, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \csc^6(e+fx)(a\sin(e+fx)+a)^2(c-c\sin(e+fx))\,dx$$
$$\Big\downarrow \ 3042$$
$$\int \frac{(a\sin(e+fx)+a)^2(c-c\sin(e+fx))}{\sin(e+fx)^6}dx$$
$$\Big\downarrow \ 3445$$

$$\int \left(a^2c\csc^6(e+fx)+a^2c\csc^5(e+fx)-a^2c\csc^4(e+fx)-a^2c\csc^3(e+fx)\right)dx$$

$$\Big\downarrow \text{2009}$$

$$\frac{a^2c\operatorname{arctanh}(\cos(e+fx))}{8f}-\frac{a^2c\cot^5(e+fx)}{5f}-\frac{a^2c\cot^3(e+fx)}{3f}-\frac{a^2c\cot(e+fx)\csc^3(e+fx)}{4f}+\frac{a^2c\cot(e+fx)\csc(e+fx)}{8f}$$

input
```
Int[Csc[e + f*x]^6*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output
```
(a^2*c*ArcTanh[Cos[e + f*x]])/(8*f) - (a^2*c*Cot[e + f*x]^3)/(3*f) - (a^2*
c*Cot[e + f*x]^5)/(5*f) + (a^2*c*Cot[e + f*x]*Csc[e + f*x])/(8*f) - (a^2*c
*Cot[e + f*x]*Csc[e + f*x]^3)/(4*f)
```

3.10.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3445
```
Int[sin[(e_.) + (f_.)*(x_)]^(n_.)*((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m
_.)*((A_.) + (B_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] :> Int[ExpandTrig[si
n[e + f*x]^n*(a + b*sin[e + f*x])^m*(A + B*sin[e + f*x]), x], x] /; FreeQ[{
a, b, e, f, A, B}, x] && EqQ[A*b + a*B, 0] && EqQ[a^2 - b^2, 0] && IntegerQ
[m] && IntegerQ[n]
```

3.10.4 Maple [A] (verified)

Time = 1.24 (sec) , antiderivative size = 123, normalized size of antiderivative = 1.17

method	result
parallelrisch	$-\frac{a^2c\left(-6\left(\tan^5\left(\frac{fx}{2}+\frac{e}{2}\right)\right)+6\left(\cot^5\left(\frac{fx}{2}+\frac{e}{2}\right)\right)-15\left(\tan^4\left(\frac{fx}{2}+\frac{e}{2}\right)\right)+15\left(\cot^4\left(\frac{fx}{2}+\frac{e}{2}\right)\right)-10\left(\tan^3\left(\frac{fx}{2}+\frac{e}{2}\right)\right)+10\left(\cot^3\right.\right.}{960f}$
risch	$-\frac{a^2c\left(15\,e^{9i(fx+e)}+240ie^{6i(fx+e)}+90\,e^{7i(fx+e)}+80ie^{4i(fx+e)}+80ie^{2i(fx+e)}-90\,e^{3i(fx+e)}-16i-15\,e^{i(fx+e)}\right)}{60f\left(e^{2i(fx+e)}-1\right)^5}-\frac{a^2c}{}$
derivativedivides	$\frac{-a^2c\left(-\frac{\csc(fx+e)\cot(fx+e)}{2}+\frac{\ln(\csc(fx+e)-\cot(fx+e))}{2}\right)-a^2c\left(-\frac{2}{3}-\frac{\left(\csc^2(fx+e)\right)}{3}\right)\cot(fx+e)+a^2c\left(\left(-\frac{\left(\csc^3(fx+e)\right)}{4}-\right.\right.}{f}$
default	$\frac{-a^2c\left(-\frac{\csc(fx+e)\cot(fx+e)}{2}+\frac{\ln(\csc(fx+e)-\cot(fx+e))}{2}\right)-a^2c\left(-\frac{2}{3}-\frac{\left(\csc^2(fx+e)\right)}{3}\right)\cot(fx+e)+a^2c\left(\left(-\frac{\left(\csc^3(fx+e)\right)}{4}-\right.\right.}{f}$
norman	$\frac{-\frac{a^2c}{160f}-\frac{a^2c\left(\tan^9\left(\frac{fx}{2}+\frac{e}{2}\right)\right)}{8f}-\frac{3a^2c\left(\tan^5\left(\frac{fx}{2}+\frac{e}{2}\right)\right)}{32f}-\frac{5a^2c\left(\tan^7\left(\frac{fx}{2}+\frac{e}{2}\right)\right)}{32f}-\frac{a^2c\tan\left(\frac{fx}{2}+\frac{e}{2}\right)}{64f}-\frac{7a^2c\left(\tan^2\left(\frac{fx}{2}+\frac{e}{2}\right)\right)}{240f}-\frac{3a^2c\left(\tan\right.}{}}{}$

input
```
int(csc(f*x+e)^6*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x,method=_RETURNVERBO
SE)
```

output
```
-1/960*a^2*c*(-6*tan(1/2*f*x+1/2*e)^5+6*cot(1/2*f*x+1/2*e)^5-15*tan(1/2*f*
x+1/2*e)^4+15*cot(1/2*f*x+1/2*e)^4-10*tan(1/2*f*x+1/2*e)^3+10*cot(1/2*f*x+
1/2*e)^3+120*ln(tan(1/2*f*x+1/2*e))+60*tan(1/2*f*x+1/2*e)-60*cot(1/2*f*x+1
/2*e))/f
```

3.10.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 201 vs. 2(95) = 190.

Time = 0.27 (sec) , antiderivative size = 201, normalized size of antiderivative = 1.91

$$\int \csc^6(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$=\frac{32\,a^2c\cos\left(fx+e\right)^5-80\,a^2c\cos\left(fx+e\right)^3+15\,\left(a^2c\cos\left(fx+e\right)^4-2\,a^2c\cos\left(fx+e\right)^2+a^2c\right)\log\left(\frac{1}{2}\,\cos\left(\right.\right.}{}$$

input
```
integrate(csc(f*x+e)^6*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="f
ricas")
```

output
```
1/240*(32*a^2*c*cos(f*x + e)^5 - 80*a^2*c*cos(f*x + e)^3 + 15*(a^2*c*cos(f
*x + e)^4 - 2*a^2*c*cos(f*x + e)^2 + a^2*c)*log(1/2*cos(f*x + e) + 1/2)*si
n(f*x + e) - 15*(a^2*c*cos(f*x + e)^4 - 2*a^2*c*cos(f*x + e)^2 + a^2*c)*lo
g(-1/2*cos(f*x + e) + 1/2)*sin(f*x + e) - 30*(a^2*c*cos(f*x + e)^3 + a^2*c
*cos(f*x + e))*sin(f*x + e))/((f*cos(f*x + e)^4 - 2*f*cos(f*x + e)^2 + f)*
sin(f*x + e))
```

3.10.6 Sympy [F(-1)]

Timed out.

$$\int \csc^6(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx = \text{Timed out}$$

input
```
integrate(csc(f*x+e)**6*(a+a*sin(f*x+e))**2*(c-c*sin(f*x+e)),x)
```

output
```
Timed out
```

3.10.7 Maxima [A] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 187, normalized size of antiderivative = 1.78

$$\int \csc^6(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$= \frac{15\,a^2c\left(\frac{2\left(3\,\cos(fx+e)^3-5\,\cos(fx+e)\right)}{\cos(fx+e)^4-2\,\cos(fx+e)^2+1}-3\,\log\left(\cos\left(fx+e\right)+1\right)+3\,\log\left(\cos\left(fx+e\right)-1\right)\right)-60\,a^2c\left(\frac{2\,\cos(fx+}{\cos(fx+e)}\right.}{}$$

input
```
integrate(csc(f*x+e)^6*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="m
axima")
```

output
```
1/240*(15*a^2*c*(2*(3*cos(f*x + e)^3 - 5*cos(f*x + e))/(cos(f*x + e)^4 - 2
*cos(f*x + e)^2 + 1) - 3*log(cos(f*x + e) + 1) + 3*log(cos(f*x + e) - 1))
- 60*a^2*c*(2*cos(f*x + e)/(cos(f*x + e)^2 - 1) - log(cos(f*x + e) + 1) +
log(cos(f*x + e) - 1)) + 80*(3*tan(f*x + e)^2 + 1)*a^2*c/tan(f*x + e)^3 -
16*(15*tan(f*x + e)^4 + 10*tan(f*x + e)^2 + 3)*a^2*c/tan(f*x + e)^5)/f
```

3.10.8 Giac [A] (verification not implemented)

Time = 0.33 (sec) , antiderivative size = 174, normalized size of antiderivative = 1.66

$$\int \csc^6(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$

$$= \frac{6\,a^2c\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right)^5+15\,a^2c\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right)^4+10\,a^2c\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right)^3-120\,a^2c\log\left(\left|\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\cdots\right.\right.\right.}{9\cdots}$$

input

```
integrate(csc(f*x+e)^6*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="g
iac")
```

output

```
1/960*(6*a^2*c*tan(1/2*f*x + 1/2*e)^5 + 15*a^2*c*tan(1/2*f*x + 1/2*e)^4 +
10*a^2*c*tan(1/2*f*x + 1/2*e)^3 - 120*a^2*c*log(abs(tan(1/2*f*x + 1/2*e)))
 - 60*a^2*c*tan(1/2*f*x + 1/2*e) + (274*a^2*c*tan(1/2*f*x + 1/2*e)^5 + 60*
a^2*c*tan(1/2*f*x + 1/2*e)^4 - 10*a^2*c*tan(1/2*f*x + 1/2*e)^2 - 15*a^2*c*
tan(1/2*f*x + 1/2*e) - 6*a^2*c)/tan(1/2*f*x + 1/2*e)^5)/f
```

3.10.9 Mupad [B] (verification not implemented)

Time = 12.00 (sec) , antiderivative size = 244, normalized size of antiderivative = 2.32

$$\int \csc^6(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx =$$

$$-\frac{a^2\,c\left(6\cos\left(\frac{e}{2}+\frac{f\,x}{2}\right)^{10}-6\sin\left(\frac{e}{2}+\frac{f\,x}{2}\right)^{10}-15\,\cos\left(\frac{e}{2}+\frac{f\,x}{2}\right)\,\sin\left(\frac{e}{2}+\frac{f\,x}{2}\right)^9+15\cos\left(\frac{e}{2}+\frac{f\,x}{2}\right)^9\,\sin\left(\frac{e}{2}+\cdots\right.\right.}{\cdots}$$

input

```
int(((a + a*sin(e + f*x))^2*(c - c*sin(e + f*x)))/sin(e + f*x)^6,x)
```

output

```
-(a^2*c*(6*cos(e/2 + (f*x)/2)^10 - 6*sin(e/2 + (f*x)/2)^10 - 15*cos(e/2 +
(f*x)/2)*sin(e/2 + (f*x)/2)^9 + 15*cos(e/2 + (f*x)/2)^9*sin(e/2 + (f*x)/2)
 - 10*cos(e/2 + (f*x)/2)^2*sin(e/2 + (f*x)/2)^8 + 60*cos(e/2 + (f*x)/2)^4*
sin(e/2 + (f*x)/2)^6 - 60*cos(e/2 + (f*x)/2)^6*sin(e/2 + (f*x)/2)^4 + 10*c
os(e/2 + (f*x)/2)^8*sin(e/2 + (f*x)/2)^2 + 120*cos(e/2 + (f*x)/2)^5*log(si
n(e/2 + (f*x)/2)/cos(e/2 + (f*x)/2))*sin(e/2 + (f*x)/2)^5))/(960*f*cos(e/2
 + (f*x)/2)^5*sin(e/2 + (f*x)/2)^5)
```

3.11 $\int \csc^7(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$

3.11.1 Optimal result

Integrand size = 32, antiderivative size = 130

$$\begin{aligned}
&\int \csc^7(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx \\
&= \frac{a^2 c\,\text{arctanh}(\cos(e+fx))}{16f} - \frac{a^2 c\cot^3(e+fx)}{3f} \\
&\quad - \frac{a^2 c\cot^5(e+fx)}{5f} + \frac{a^2 c\cot(e+fx)\csc(e+fx)}{16f} \\
&\quad + \frac{a^2 c\cot(e+fx)\csc^3(e+fx)}{24f} - \frac{a^2 c\cot(e+fx)\csc^5(e+fx)}{6f}
\end{aligned}$$

output
```
1/16*a^2*c*arctanh(cos(f*x+e))/f-1/3*a^2*c*cot(f*x+e)^3/f-1/5*a^2*c*cot(f*
x+e)^5/f+1/16*a^2*c*cot(f*x+e)*csc(f*x+e)/f+1/24*a^2*c*cot(f*x+e)*csc(f*x+
e)^3/f-1/6*a^2*c*cot(f*x+e)*csc(f*x+e)^5/f
```

3.11.2 Mathematica [A] (verified)

Time = 0.12 (sec) , antiderivative size = 204, normalized size of antiderivative = 1.57

$$\int \csc^7(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$=\frac{2a^2c\cot(e+fx)}{15f}+\frac{a^2c\csc^2\left(\frac{1}{2}(e+fx)\right)}{64f}-\frac{a^2c\csc^6\left(\frac{1}{2}(e+fx)\right)}{384f}+\frac{a^2c\cot(e+fx)\csc^2(e+fx)}{15f}-\frac{a^2c\cot(e+fx)\csc^4(e+fx)}{5f}+\frac{a^2c\log\left(\cos\left(\frac{1}{2}(e+fx)\right)\right)}{16f}-\frac{a^2c\log\left(\sin\left(\frac{1}{2}(e+fx)\right)\right)}{16f}-\frac{a^2c\sec^2\left(\frac{1}{2}(e+fx)\right)}{64f}+\frac{a^2c\sec^6\left(\frac{1}{2}(e+fx)\right)}{384f}$$

input
```
Integrate[Csc[e + f*x]^7*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output
```
(2*a^2*c*Cot[e + f*x])/(15*f) + (a^2*c*Csc[(e + f*x)/2]^2)/(64*f) - (a^2*c
*Csc[(e + f*x)/2]^6)/(384*f) + (a^2*c*Cot[e + f*x]*Csc[e + f*x]^2)/(15*f)
- (a^2*c*Cot[e + f*x]*Csc[e + f*x]^4)/(5*f) + (a^2*c*Log[Cos[(e + f*x)/2]]
)/(16*f) - (a^2*c*Log[Sin[(e + f*x)/2]])/(16*f) - (a^2*c*Sec[(e + f*x)/2]^
2)/(64*f) + (a^2*c*Sec[(e + f*x)/2]^6)/(384*f)
```

3.11.3 Rubi [A] (verified)

Time = 0.40 (sec) , antiderivative size = 130, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.094, Rules used = {3042, 3445, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \csc^7(e+fx)(a\sin(e+fx)+a)^2(c-c\sin(e+fx))\,dx$$
$$\Big\downarrow \text{3042}$$
$$\int \frac{(a\sin(e+fx)+a)^2(c-c\sin(e+fx))}{\sin(e+fx)^7}dx$$
$$\Big\downarrow \text{3445}$$

$$\int \left(a^2 c \csc^7(e+fx)+a^2 c \csc^6(e+fx)-a^2 c \csc^5(e+fx)-a^2 c \csc^4(e+fx)\right) dx$$

$$\Big\downarrow \text{2009}$$

$$\frac{a^2 c \text{arctanh}(\cos(e+fx))}{16f}-\frac{a^2 c \cot^5(e+fx)}{5f}-\frac{a^2 c \cot^3(e+fx)}{3f}-\frac{a^2 c \cot(e+fx)\csc^5(e+fx)}{6f}+\frac{a^2 c \cot(e+fx)\csc^3(e+fx)}{24f}+\frac{a^2 c \cot(e+fx)\csc(e+fx)}{16f}$$

input
```
Int[Csc[e + f*x]^7*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output
```
(a^2*c*ArcTanh[Cos[e + f*x]])/(16*f) - (a^2*c*Cot[e + f*x]^3)/(3*f) - (a^2
*c*Cot[e + f*x]^5)/(5*f) + (a^2*c*Cot[e + f*x]*Csc[e + f*x])/(16*f) + (a^2
*c*Cot[e + f*x]*Csc[e + f*x]^3)/(24*f) - (a^2*c*Cot[e + f*x]*Csc[e + f*x]^
5)/(6*f)
```

3.11.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3445
```
Int[sin[(e_.) + (f_.)*(x_)]^(n_.)*((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m
_.)*((A_.) + (B_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] :> Int[ExpandTrig[si
n[e + f*x]^n*(a + b*sin[e + f*x])^m*(A + B*sin[e + f*x]), x], x] /; FreeQ[{
a, b, e, f, A, B}, x] && EqQ[A*b + a*B, 0] && EqQ[a^2 - b^2, 0] && IntegerQ
[m] && IntegerQ[n]
```

3.11.4 Maple [A] (verified)

Time = 1.33 (sec) , antiderivative size = 125, normalized size of antiderivative = 0.96

method	result
parallelrisch	$-\frac{19c\left(\frac{512\ln\left(\tan\left(\frac{fx}{2}+\frac{e}{2}\right)\right)}{19}+\left(\sec\left(\frac{fx}{2}+\frac{e}{2}\right)\left(\cos(fx+e)+\frac{17\cos(3fx+3e)}{114}-\frac{\cos(5fx+5e)}{38}\right)\csc\left(\frac{fx}{2}+\frac{e}{2}\right)+\frac{128\cos(fx+e)}{57}+\frac{32}{}\right.}{8192f}$
risch	$-\frac{a^2c\left(15\,\mathrm{e}^{11i(fx+e)}-85\,\mathrm{e}^{9i(fx+e)}-570\,\mathrm{e}^{7i(fx+e)}+480i\mathrm{e}^{8i(fx+e)}-570\,\mathrm{e}^{5i(fx+e)}-320i\mathrm{e}^{6i(fx+e)}-85\,\mathrm{e}^{3i(fx+e)}+15\,\mathrm{e}^{i(fx}\right.}{120f\left(\mathrm{e}^{2i(fx+e)}-1\right)^6}$
derivativedivides	$\frac{-a^2c\left(-\frac{2}{3}-\frac{\left(\csc^2(fx+e)\right)}{3}\right)\cot(fx+e)-a^2c\left(\left(\left(-\frac{\left(\csc^3(fx+e)\right)}{4}-\frac{3\csc(fx+e)}{8}\right)\cot(fx+e)+\frac{3\ln(\csc(fx+e)-\cot(fx+e))}{8}\right)+\right.}{}$
default	$\frac{-a^2c\left(-\frac{2}{3}-\frac{\left(\csc^2(fx+e)\right)}{3}\right)\cot(fx+e)-a^2c\left(\left(\left(-\frac{\left(\csc^3(fx+e)\right)}{4}-\frac{3\csc(fx+e)}{8}\right)\cot(fx+e)+\frac{3\ln(\csc(fx+e)-\cot(fx+e))}{8}\right)+\right.}{}$

input
```
int(csc(f*x+e)^7*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x,method=_RETURNVERBO
SE)
```

output
```
-19/8192*c*(512/19*ln(tan(1/2*f*x+1/2*e))+(sec(1/2*f*x+1/2*e)*(cos(f*x+e)+
17/114*cos(3*f*x+3*e)-1/38*cos(5*f*x+5*e))*csc(1/2*f*x+1/2*e)+128/57*cos(f
*x+e)+32/57*cos(3*f*x+3*e)-32/285*cos(5*f*x+5*e))*sec(1/2*f*x+1/2*e)^5*csc
(1/2*f*x+1/2*e)^5)*a^2/f
```

3.11.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 240 vs. 2(118) = 236.

Time = 0.27 (sec) , antiderivative size = 240, normalized size of antiderivative = 1.85

$$\int \csc^7(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx =$$
$$-\frac{30\,a^2c\cos\left(fx+e\right)^5-80\,a^2c\cos\left(fx+e\right)^3-30\,a^2c\cos\left(fx+e\right)-15\left(a^2c\cos\left(fx+e\right)^6-3\,a^2c\cos\left(fx\right.\right.}{}$$

input
```
integrate(csc(f*x+e)^7*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="f
ricas")
```

output
```
-1/480*(30*a^2*c*cos(f*x + e)^5 - 80*a^2*c*cos(f*x + e)^3 - 30*a^2*c*cos(f
*x + e) - 15*(a^2*c*cos(f*x + e)^6 - 3*a^2*c*cos(f*x + e)^4 + 3*a^2*c*cos(
f*x + e)^2 - a^2*c)*log(1/2*cos(f*x + e) + 1/2) + 15*(a^2*c*cos(f*x + e)^6
 - 3*a^2*c*cos(f*x + e)^4 + 3*a^2*c*cos(f*x + e)^2 - a^2*c)*log(-1/2*cos(f
*x + e) + 1/2) + 32*(2*a^2*c*cos(f*x + e)^5 - 5*a^2*c*cos(f*x + e)^3)*sin(
f*x + e))/(f*cos(f*x + e)^6 - 3*f*cos(f*x + e)^4 + 3*f*cos(f*x + e)^2 - f)
```

3.11.6 Sympy [F(-1)]

Timed out.

$$\int \csc^7(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx = \text{Timed out}$$

input
```
integrate(csc(f*x+e)**7*(a+a*sin(f*x+e))**2*(c-c*sin(f*x+e)),x)
```

output
```
Timed out
```

3.11.7 Maxima [A] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 232, normalized size of antiderivative = 1.78

$$\int \csc^7(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$= \frac{5\,a^2c\left(\frac{2\left(15\,\cos(fx+e)^5-40\,\cos(fx+e)^3+33\,\cos(fx+e)\right)}{\cos(fx+e)^6-3\,\cos(fx+e)^4+3\,\cos(fx+e)^2-1}-15\,\log\left(\cos\left(fx+e\right)+1\right)+15\,\log\left(\cos\left(fx+e\right)-1\right)\right)-3}{}$$

input
```
integrate(csc(f*x+e)^7*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="m
axima")
```

output
```
1/480*(5*a^2*c*(2*(15*cos(f*x + e)^5 - 40*cos(f*x + e)^3 + 33*cos(f*x + e)
)/(cos(f*x + e)^6 - 3*cos(f*x + e)^4 + 3*cos(f*x + e)^2 - 1) - 15*log(cos(
f*x + e) + 1) + 15*log(cos(f*x + e) - 1)) - 30*a^2*c*(2*(3*cos(f*x + e)^3
- 5*cos(f*x + e))/(cos(f*x + e)^4 - 2*cos(f*x + e)^2 + 1) - 3*log(cos(f*x
+ e) + 1) + 3*log(cos(f*x + e) - 1)) + 160*(3*tan(f*x + e)^2 + 1)*a^2*c/ta
n(f*x + e)^3 - 32*(15*tan(f*x + e)^4 + 10*tan(f*x + e)^2 + 3)*a^2*c/tan(f*
x + e)^5)/f
```

3.11.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 242 vs. 2(118) = 236.

Time = 0.38 (sec) , antiderivative size = 242, normalized size of antiderivative = 1.86

$$\int \csc^7(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$$
$$= \frac{5\,a^2c\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right)^6+12\,a^2c\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right)^5+15\,a^2c\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right)^4+20\,a^2c\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right)^3-}{}$$

input
```
integrate(csc(f*x+e)^7*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="g
iac")
```

output
```
1/1920*(5*a^2*c*tan(1/2*f*x + 1/2*e)^6 + 12*a^2*c*tan(1/2*f*x + 1/2*e)^5 +
 15*a^2*c*tan(1/2*f*x + 1/2*e)^4 + 20*a^2*c*tan(1/2*f*x + 1/2*e)^3 - 15*a^
2*c*tan(1/2*f*x + 1/2*e)^2 - 120*a^2*c*log(abs(tan(1/2*f*x + 1/2*e))) - 12
0*a^2*c*tan(1/2*f*x + 1/2*e) + (294*a^2*c*tan(1/2*f*x + 1/2*e)^6 + 120*a^2
*c*tan(1/2*f*x + 1/2*e)^5 + 15*a^2*c*tan(1/2*f*x + 1/2*e)^4 - 20*a^2*c*tan
(1/2*f*x + 1/2*e)^3 - 15*a^2*c*tan(1/2*f*x + 1/2*e)^2 - 12*a^2*c*tan(1/2*f
*x + 1/2*e) - 5*a^2*c)/tan(1/2*f*x + 1/2*e)^6)/f
```

3.11.9 Mupad [B] (verification not implemented)

Time = 12.23 (sec) , antiderivative size = 340, normalized size of antiderivative = 2.62

$$\int \csc^7(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx =$$
$$-\frac{a^2\,c\left(5\cos\left(\frac{e}{2}+\frac{f\,x}{2}\right)^{12}-5\sin\left(\frac{e}{2}+\frac{f\,x}{2}\right)^{12}-12\,\cos\left(\frac{e}{2}+\frac{f\,x}{2}\right)\,\sin\left(\frac{e}{2}+\frac{f\,x}{2}\right)^{11}+12\cos\left(\frac{e}{2}+\frac{f\,x}{2}\right)^{11}\,\sin\left(\frac{e}{2}\right.\right.}{}$$

input
```
int(((a + a*sin(e + f*x))^2*(c - c*sin(e + f*x)))/sin(e + f*x)^7,x)
```

output
```
-(a^2*c*(5*cos(e/2 + (f*x)/2)^12 - 5*sin(e/2 + (f*x)/2)^12 - 12*cos(e/2 +
(f*x)/2)*sin(e/2 + (f*x)/2)^11 + 12*cos(e/2 + (f*x)/2)^11*sin(e/2 + (f*x)/
2) - 15*cos(e/2 + (f*x)/2)^2*sin(e/2 + (f*x)/2)^10 - 20*cos(e/2 + (f*x)/2)
^3*sin(e/2 + (f*x)/2)^9 + 15*cos(e/2 + (f*x)/2)^4*sin(e/2 + (f*x)/2)^8 + 1
20*cos(e/2 + (f*x)/2)^5*sin(e/2 + (f*x)/2)^7 - 120*cos(e/2 + (f*x)/2)^7*si
n(e/2 + (f*x)/2)^5 - 15*cos(e/2 + (f*x)/2)^8*sin(e/2 + (f*x)/2)^4 + 20*cos
(e/2 + (f*x)/2)^9*sin(e/2 + (f*x)/2)^3 + 15*cos(e/2 + (f*x)/2)^10*sin(e/2
+ (f*x)/2)^2 + 120*cos(e/2 + (f*x)/2)^6*log(sin(e/2 + (f*x)/2)/cos(e/2 + (
f*x)/2))*sin(e/2 + (f*x)/2)^6))/(1920*f*cos(e/2 + (f*x)/2)^6*sin(e/2 + (f*
x)/2)^6)
```

3.11. $\int \csc^7(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))\,dx$

3.12 $\int \sin^2(c+dx)(a+a\sin(c+dx))^{3/2}(c-c\sin(c+dx))\,dx$

3.12.1 Optimal result

Integrand size = 34, antiderivative size = 128

$$\int \sin^2(c+dx)(a+a\sin(c+dx))^{3/2}(c-c\sin(c+dx))\,dx = -\frac{8a^3c\cos^3(c+dx)}{63d(a+a\sin(c+dx))^{3/2}}-\frac{2a^2c\cos^3(c+dx)}{21d\sqrt{a+a\sin(c+dx)}}+\frac{4ac\cos^3(c+dx)\sqrt{a+a\sin(c+dx)}}{21d}-\frac{2c\cos^3(c+dx)(a+a\sin(c+dx))^{3/2}}{9d}$$

output
```
-8/63*a^3*c*cos(d*x+c)^3/d/(a+a*sin(d*x+c))^(3/2)-2/9*c*cos(d*x+c)^3*(a+a*
sin(d*x+c))^(3/2)/d-2/21*a^2*c*cos(d*x+c)^3/d/(a+a*sin(d*x+c))^(1/2)+4/21*
a*c*cos(d*x+c)^3*(a+a*sin(d*x+c))^(1/2)/d
```

3.12.2 Mathematica [A] (verified)

Time = 1.75 (sec) , antiderivative size = 69, normalized size of antiderivative = 0.54

$$\int \sin^2(c+dx)(a+a\sin(c+dx))^{3/2}(c-c\sin(c+dx))\,dx = -\frac{2ac\sec(c+dx)(-1+\sin(c+dx))^2\sqrt{a(1+\sin(c+dx))}(8+12\sin(c+dx)+15\sin^2(c+dx)+7\sin^3(c+}{63d}$$

input
```
Integrate[Sin[c + d*x]^2*(a + a*Sin[c + d*x])^(3/2)*(c - c*Sin[c + d*x]),x
]
```

output
```
(-2*a*c*Sec[c + d*x]*(-1 + Sin[c + d*x])^2*Sqrt[a*(1 + Sin[c + d*x])]*(8 +
 12*Sin[c + d*x] + 15*Sin[c + d*x]^2 + 7*Sin[c + d*x]^3))/(63*d)
```

3.12.3 Rubi [A] (verified)

Time = 0.90 (sec) , antiderivative size = 185, normalized size of antiderivative = 1.45, number of steps used = 12, number of rules used = 12, $\frac{\text{number of rules}}{\text{integrand size}} = 0.353$, Rules used = {3042, 3455, 27, 3042, 3460, 3042, 3238, 27, 3042, 3230, 3042, 3125}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \sin^2(c+dx)(c-c\sin(c+dx))(a\sin(c+dx)+a)^{3/2}\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int \sin(c+dx)^2(c-c\sin(c+dx))(a\sin(c+dx)+a)^{3/2}dx$$

$$\Big\downarrow \text{3455}$$

$$\frac{2}{9}\int \frac{1}{2}\sin^2(c+dx)\sqrt{\sin(c+dx)a+a}(3ac-ac\sin(c+dx))dx+\frac{2ac\sin^3(c+dx)\cos(c+dx)\sqrt{a\sin(c+dx)+a}}{9d}$$

$$\Big\downarrow \text{27}$$

$$\frac{1}{9}\int \sin^2(c+dx)\sqrt{\sin(c+dx)a+a}(3ac-ac\sin(c+dx))dx+\frac{2ac\sin^3(c+dx)\cos(c+dx)\sqrt{a\sin(c+dx)+a}}{9d}$$

$$\Big\downarrow \text{3042}$$

$$\frac{1}{9}\int \sin(c+dx)^2\sqrt{\sin(c+dx)a+a}(3ac-ac\sin(c+dx))dx+\frac{2ac\sin^3(c+dx)\cos(c+dx)\sqrt{a\sin(c+dx)+a}}{9d}$$

$$\Big\downarrow \text{3460}$$

$$\frac{1}{9}\left(\frac{15}{7}ac\int \sin^2(c+dx)\sqrt{\sin(c+dx)a+a}dx+\frac{2a^2c\sin^3(c+dx)\cos(c+dx)}{7d\sqrt{a\sin(c+dx)+a}}\right)+\frac{2ac\sin^3(c+dx)\cos(c+dx)\sqrt{a\sin(c+dx)+a}}{9d}$$

$\big\downarrow$ 3042

$$\frac{1}{9}\left(\frac{15}{7}ac\int \sin(c+dx)^2\sqrt{\sin(c+dx)a+a}dx+\frac{2a^2c\sin^3(c+dx)\cos(c+dx)}{7d\sqrt{a\sin(c+dx)+a}}\right)+\frac{2ac\sin^3(c+dx)\cos(c+dx)\sqrt{a\sin(c+dx)+a}}{9d}$$

$\big\downarrow$ 3238

$$\frac{1}{9}\left(\frac{15}{7}ac\left(\frac{2\int \frac{1}{2}(3a-2a\sin(c+dx))\sqrt{\sin(c+dx)a+a}dx}{5a}-\frac{2\cos(c+dx)(a\sin(c+dx)+a)^{3/2}}{5ad}\right)+\frac{2a^2c\sin^3(c-}{7d\sqrt{a\sin}}\right.$$
$$\frac{2ac\sin^3(c+dx)\cos(c+dx)\sqrt{a\sin(c+dx)+a}}{9d}$$

$\big\downarrow$ 27

$$\frac{1}{9}\left(\frac{15}{7}ac\left(\frac{\int (3a-2a\sin(c+dx))\sqrt{\sin(c+dx)a+a}dx}{5a}-\frac{2\cos(c+dx)(a\sin(c+dx)+a)^{3/2}}{5ad}\right)+\frac{2a^2c\sin^3(c+d}{7d\sqrt{a\sin(c}}\right.$$
$$\frac{2ac\sin^3(c+dx)\cos(c+dx)\sqrt{a\sin(c+dx)+a}}{9d}$$

$\big\downarrow$ 3042

$$\frac{1}{9}\left(\frac{15}{7}ac\left(\frac{\int (3a-2a\sin(c+dx))\sqrt{\sin(c+dx)a+a}dx}{5a}-\frac{2\cos(c+dx)(a\sin(c+dx)+a)^{3/2}}{5ad}\right)+\frac{2a^2c\sin^3(c+d}{7d\sqrt{a\sin(c}}\right.$$
$$\frac{2ac\sin^3(c+dx)\cos(c+dx)\sqrt{a\sin(c+dx)+a}}{9d}$$

$\big\downarrow$ 3230

$$\frac{1}{9}\left(\frac{15}{7}ac\left(\frac{\frac{7}{3}a\int \sqrt{\sin(c+dx)a+a}dx+\frac{4a\cos(c+dx)\sqrt{a\sin(c+dx)+a}}{3d}}{5a}-\frac{2\cos(c+dx)(a\sin(c+dx)+a)^{3/2}}{5ad}\right)+\frac{2a^2c\,s}{7d}\right.$$
$$\frac{2ac\sin^3(c+dx)\cos(c+dx)\sqrt{a\sin(c+dx)+a}}{9d}$$

$\big\downarrow$ 3042

3.12. $\int \sin^2(c+dx)(a+a\sin(c+dx))^{3/2}(c-c\sin(c+dx))\,dx$

$$\frac{1}{9}\left(\frac{15}{7}ac\left(\frac{\frac{7}{3}a\int \sqrt{\sin(c+dx)a+a}dx+\frac{4a\cos(c+dx)\sqrt{a\sin(c+dx)+a}}{3d}}{5a}-\frac{2\cos(c+dx)(a\sin(c+dx)+a)^{3/2}}{5ad}\right)+\frac{2a^2c\,\mathrm{s}}{7c}\right.$$

$$\frac{2ac\sin^3(c+dx)\cos(c+dx)\sqrt{a\sin(c+dx)+a}}{9d}$$

$$\Big\downarrow \; 3125$$

$$\frac{1}{9}\left(\frac{2a^2c\sin^3(c+dx)\cos(c+dx)}{7d\sqrt{a\sin(c+dx)+a}}+\frac{15}{7}ac\left(\frac{\frac{4a\cos(c+dx)\sqrt{a\sin(c+dx)+a}}{3d}-\frac{14a^2\cos(c+dx)}{3d\sqrt{a\sin(c+dx)+a}}}{5a}-\frac{2\cos(c+dx)(a\sin(c+a}{5ad}\right.\right.$$

$$\frac{2ac\sin^3(c+dx)\cos(c+dx)\sqrt{a\sin(c+dx)+a}}{9d}$$

input
```
Int[Sin[c + d*x]^2*(a + a*Sin[c + d*x])^(3/2)*(c - c*Sin[c + d*x]),x]
```

output
```
(2*a*c*Cos[c + d*x]*Sin[c + d*x]^3*Sqrt[a + a*Sin[c + d*x]])/(9*d) + ((2*a
^2*c*Cos[c + d*x]*Sin[c + d*x]^3)/(7*d*Sqrt[a + a*Sin[c + d*x]]) + (15*a*c
*((-2*Cos[c + d*x]*(a + a*Sin[c + d*x])^(3/2))/(5*a*d) + ((-14*a^2*Cos[c +
 d*x])/(3*d*Sqrt[a + a*Sin[c + d*x]]) + (4*a*Cos[c + d*x]*Sqrt[a + a*Sin[c
 + d*x]])/(3*d))/(5*a)))/7)/9
```

3.12.3.1 Defintions of rubi rules used

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3125
```
Int[Sqrt[(a_) + (b_.)*sin[(c_.) + (d_.)*(x_)]], x_Symbol] :> Simp[-2*b*(Cos
[c + d*x]/(d*Sqrt[a + b*Sin[c + d*x]])), x] /; FreeQ[{a, b, c, d}, x] && Eq
Q[a^2 - b^2, 0]
```

rule 3230
```
Int[((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_)*((c_.) + (d_.)*sin[(e_.) +
(f_.)*(x_)]), x_Symbol] :> Simp[(-d)*Cos[e + f*x]*((a + b*Sin[e + f*x])^m/(
f*(m + 1))), x] + Simp[(a*d*m + b*c*(m + 1))/(b*(m + 1))   Int[(a + b*Sin[e
 + f*x])^m, x], x] /; FreeQ[{a, b, c, d, e, f, m}, x] && NeQ[b*c - a*d, 0]
&& EqQ[a^2 - b^2, 0] &&  !LtQ[m, -2^(-1)]
```

rule 3238
```
Int[sin[(e_.) + (f_.)*(x_)]^2*((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_),
x_Symbol] :> Simp[(-Cos[e + f*x])*((a + b*Sin[e + f*x])^(m + 1)/(b*f*(m + 2
))), x] + Simp[1/(b*(m + 2))   Int[(a + b*Sin[e + f*x])^m*(b*(m + 1) - a*Si
n[e + f*x]), x], x] /; FreeQ[{a, b, e, f, m}, x] && EqQ[a^2 - b^2, 0] &&  !
LtQ[m, -2^(-1)]
```

rule 3455
```
Int[((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_)*((A_.) + (B_.)*sin[(e_.) +
(f_.)*(x_)])*((c_.) + (d_.)*sin[(e_.) + (f_.)*(x_)])^(n_), x_Symbol] :> Sim
p[(-b)*B*Cos[e + f*x]*(a + b*Sin[e + f*x])^(m - 1)*((c + d*Sin[e + f*x])^(n
 + 1)/(d*f*(m + n + 1))), x] + Simp[1/(d*(m + n + 1))   Int[(a + b*Sin[e +
f*x])^(m - 1)*(c + d*Sin[e + f*x])^n*Simp[a*A*d*(m + n + 1) + B*(a*c*(m - 1
) + b*d*(n + 1)) + (A*b*d*(m + n + 1) - B*(b*c*m - a*d*(2*m + n)))*Sin[e +
f*x], x], x], x] /; FreeQ[{a, b, c, d, e, f, A, B, n}, x] && NeQ[b*c - a*d,
 0] && EqQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0] && GtQ[m, 1/2] &&  !LtQ[n, -1
] && IntegerQ[2*m] && (IntegerQ[2*n] || EqQ[c, 0])
```

rule 3460
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]*((A_.) + (B_.)*sin[(e_.) + (
f_.)*(x_)])*((c_.) + (d_.)*sin[(e_.) + (f_.)*(x_)])^(n_), x_Symbol] :> Simp
[-2*b*B*Cos[e + f*x]*((c + d*Sin[e + f*x])^(n + 1)/(d*f*(2*n + 3)*Sqrt[a +
b*Sin[e + f*x]])), x] + Simp[(A*b*d*(2*n + 3) - B*(b*c - 2*a*d*(n + 1)))/(b
*d*(2*n + 3))   Int[Sqrt[a + b*Sin[e + f*x]]*(c + d*Sin[e + f*x])^n, x], x]
 /; FreeQ[{a, b, c, d, e, f, A, B, n}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 -
 b^2, 0] && NeQ[c^2 - d^2, 0] &&  !LtQ[n, -1]
```

3.12. $\int \sin^2(c+dx)(a+a\sin(c+dx))^{3/2}(c-c\sin(c+dx))\,dx$

3.12.4 Maple [A] (verified)

Time = 1.11 (sec) , antiderivative size = 78, normalized size of antiderivative = 0.61

method	result
default	$-\frac{2(1+\sin(dx+c))a^2(\sin(dx+c)-1)^2c(7(\sin^3(dx+c))+15(\sin^2(dx+c))+12\sin(dx+c)+8)}{63\cos(dx+c)\sqrt{a+a\sin(dx+c)}\,d}$
parts	$\frac{2c(1+\sin(dx+c))a^2(\sin(dx+c)-1)(15(\sin^3(dx+c))+39(\sin^2(dx+c))+52\sin(dx+c)+104)}{105\cos(dx+c)\sqrt{a+a\sin(dx+c)}\,d}-\frac{2c(1+\sin(dx+c))a^2(\sin(dx+c)-1)(3}{}$

input
```
int(sin(d*x+c)^2*(a+a*sin(d*x+c))^(3/2)*(c-c*sin(d*x+c)),x,method=_RETURNV
ERBOSE)
```

output
```
-2/63*(1+sin(d*x+c))*a^2*(sin(d*x+c)-1)^2*c*(7*sin(d*x+c)^3+15*sin(d*x+c)^
2+12*sin(d*x+c)+8)/cos(d*x+c)/(a+a*sin(d*x+c))^(1/2)/d
```

3.12.5 Fricas [A] (verification not implemented)

Time = 0.26 (sec) , antiderivative size = 155, normalized size of antiderivative = 1.21

$$\int \sin^2(c+dx)(a+a\sin(c+dx))^{3/2}(c-c\sin(c+dx))\,dx = \frac{2\left(7\,ac\cos\left(dx+c\right)^5-ac\cos\left(dx+c\right)^4-11\,ac\cos\left(dx+c\right)^3+ac\cos\left(dx+c\right)^2-4\,ac\,\mathrm{cc}\right.}{}$$

input
```
integrate(sin(d*x+c)^2*(a+a*sin(d*x+c))^(3/2)*(c-c*sin(d*x+c)),x, algorith
m="fricas")
```

output
```
2/63*(7*a*c*cos(d*x + c)^5 - a*c*cos(d*x + c)^4 - 11*a*c*cos(d*x + c)^3 +
a*c*cos(d*x + c)^2 - 4*a*c*cos(d*x + c) - 8*a*c - (7*a*c*cos(d*x + c)^4 +
8*a*c*cos(d*x + c)^3 - 3*a*c*cos(d*x + c)^2 - 4*a*c*cos(d*x + c) - 8*a*c)*
sin(d*x + c))*sqrt(a*sin(d*x + c) + a)/(d*cos(d*x + c) + d*sin(d*x + c) +
d)
```

3.12.6 Sympy [F]

$$\int \sin^2(c+dx)(a+a\sin(c+dx))^{3/2}(c-c\sin(c+dx))\,dx = -c\left(\int \left(-a\sqrt{a\sin(c+dx)+a}\sin^2(c+dx)\right)dx + \int a\sqrt{a\sin(c+dx)+a}\sin^4(c+dx)\,dx\right)$$

input
```
integrate(sin(d*x+c)**2*(a+a*sin(d*x+c))**(3/2)*(c-c*sin(d*x+c)),x)
```

output
```
-c*(Integral(-a*sqrt(a*sin(c + d*x) + a)*sin(c + d*x)**2, x) + Integral(a*
sqrt(a*sin(c + d*x) + a)*sin(c + d*x)**4, x))
```

3.12.7 Maxima [F]

$$\int \sin^2(c+dx)(a+a\sin(c+dx))^{3/2}(c-c\sin(c+dx))\,dx = \int -(a\sin(dx+c)+a)^{\frac{3}{2}}(c\sin(dx+c)-c)\sin(dx+c)^2\,dx$$

input
```
integrate(sin(d*x+c)^2*(a+a*sin(d*x+c))^(3/2)*(c-c*sin(d*x+c)),x, algorith
m="maxima")
```

output
```
-integrate((a*sin(d*x + c) + a)^(3/2)*(c*sin(d*x + c) - c)*sin(d*x + c)^2,
 x)
```

3.12.8 Giac [A] (verification not implemented)

Time = 0.35 (sec) , antiderivative size = 99, normalized size of antiderivative = 0.77

$$\int \sin^2(c+dx)(a+a\sin(c+dx))^{3/2}(c-c\sin(c+dx))\,dx = \frac{\sqrt{2}\left(126\,a\operatorname{csgn}\left(\cos\left(-\frac{1}{4}\pi+\frac{1}{2}dx+\frac{1}{2}c\right)\right)\sin\left(-\frac{1}{4}\pi+\frac{1}{2}dx+\frac{1}{2}c\right)-9\,a\operatorname{csgn}\left(\cos\left(-\frac{1}{4}\pi+\right.\right.\right.}{}$$

input
```
integrate(sin(d*x+c)^2*(a+a*sin(d*x+c))^(3/2)*(c-c*sin(d*x+c)),x, algorith
m="giac")
```

output
```
1/504*sqrt(2)*(126*a*c*sgn(cos(-1/4*pi + 1/2*d*x + 1/2*c))*sin(-1/4*pi + 1
/2*d*x + 1/2*c) - 9*a*c*sgn(cos(-1/4*pi + 1/2*d*x + 1/2*c))*sin(-7/4*pi +
7/2*d*x + 7/2*c) - 7*a*c*sgn(cos(-1/4*pi + 1/2*d*x + 1/2*c))*sin(-9/4*pi +
 9/2*d*x + 9/2*c))*sqrt(a)/d
```

3.12.9 Mupad [F(-1)]

Timed out.

$$\int \sin^2(c+dx)(a+a\sin(c+dx))^{3/2}(c-c\sin(c+dx))\,dx=\int \sin\left(c+d\,x\right)^2\,{\left(a+a\,\sin\left(c+d\,x\right)\right)}^{3/2}\,\left(c-c\,\sin\left(c+d\,x\right)\right)\,dx$$

input
```
int(sin(c + d*x)^2*(a + a*sin(c + d*x))^(3/2)*(c - c*sin(c + d*x)),x)
```

output
```
int(sin(c + d*x)^2*(a + a*sin(c + d*x))^(3/2)*(c - c*sin(c + d*x)), x)
```

3.13 $\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{c-c\sin(e+fx)}\,dx$

3.13.1 Optimal result

Integrand size = 34, antiderivative size = 69

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{c-c\sin(e+fx)}\,dx = -\frac{2\sqrt{a}\operatorname{arctanh}\left(\frac{\sqrt{a}\cos(e+fx)}{\sqrt{a+a\sin(e+fx)}}\right)}{cf} + \frac{2\sec(e+fx)\sqrt{a+a\sin(e+fx)}}{cf}$$

output
```
-2*arctanh(cos(f*x+e)*a^(1/2)/(a+a*sin(f*x+e))^(1/2))*a^(1/2)/c/f+2*sec(f*
x+e)*(a+a*sin(f*x+e))^(1/2)/c/f
```

3.13.2 Mathematica [C] (verified)

Result contains higher order function than in optimal. Order 5 vs. order 3 in optimal.

Time = 1.70 (sec) , antiderivative size = 46, normalized size of antiderivative = 0.67

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{c-c\sin(e+fx)}\,dx = \frac{2\,\text{Hypergeometric2F1}\left(-\frac{1}{2},1,\frac{1}{2},1-\sin(e+fx)\right)\sec(e+fx)\sqrt{a(1+\sin(e+fx))}}{cf}$$

input
```
Integrate[(Csc[e + f*x]*Sqrt[a + a*Sin[e + f*x]])/(c - c*Sin[e + f*x]),x]
```

output
```
(2*Hypergeometric2F1[-1/2, 1, 1/2, 1 - Sin[e + f*x]]*Sec[e + f*x]*Sqrt[a*(
1 + Sin[e + f*x])])/(c*f)
```

3.13.3 Rubi [A] (verified)

Time = 0.72 (sec) , antiderivative size = 69, normalized size of antiderivative = 1.00, number of steps used = 9, number of rules used = 8, $\frac{\text{number of rules}}{\text{integrand size}} = 0.235$, Rules used = {3042, 3413, 3042, 3215, 3042, 3152, 3252, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(e+fx)\sqrt{a\sin(e+fx)+a}}{c-c\sin(e+fx)}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\sqrt{a\sin(e+fx)+a}}{\sin(e+fx)(c-c\sin(e+fx))}dx$$

$$\Big\downarrow \text{3413}$$

$$\int \frac{\sqrt{\sin(e+fx)a+a}}{c-c\sin(e+fx)}dx+\frac{\int \csc(e+fx)\sqrt{\sin(e+fx)a+a}dx}{c}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)}dx}{c}+\int \frac{\sqrt{\sin(e+fx)a+a}}{c-c\sin(e+fx)}dx$$

$$\Big\downarrow \text{3215}$$

$$\frac{\int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)}dx}{c}+\frac{\int \sec^2(e+fx)(\sin(e+fx)a+a)^{3/2}dx}{ac}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)}dx}{c}+\frac{\int \frac{(\sin(e+fx)a+a)^{3/2}}{\cos(e+fx)^2}dx}{ac}$$

$$\Big\downarrow \text{3152}$$

$$\frac{\int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)}dx}{c}+\frac{2\sec(e+fx)\sqrt{a\sin(e+fx)+a}}{cf}$$

$$\Big\downarrow \text{3252}$$

$$\frac{2\sec(e+fx)\sqrt{a\sin(e+fx)+a}}{cf}-\frac{2a\int\frac{1}{a-\frac{a^2\cos^2(e+fx)}{\sin(e+fx)a+a}}d\frac{a\cos(e+fx)}{\sqrt{\sin(e+fx)a+a}}}{cf}$$

$$\Big\downarrow \ 219$$

$$\frac{2\sec(e+fx)\sqrt{a\sin(e+fx)+a}}{cf}-\frac{2\sqrt{a}\text{arctanh}\left(\frac{\sqrt{a}\cos(e+fx)}{\sqrt{a\sin(e+fx)+a}}\right)}{cf}$$

input
```
Int[(Csc[e + f*x]*Sqrt[a + a*Sin[e + f*x]])/(c - c*Sin[e + f*x]),x]
```

output
```
(-2*Sqrt[a]*ArcTanh[(Sqrt[a]*Cos[e + f*x])/Sqrt[a + a*Sin[e + f*x]]])/(c*f
) + (2*Sec[e + f*x]*Sqrt[a + a*Sin[e + f*x]])/(c*f)
```

3.13.3.1 Defintions of rubi rules used

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3152
```
Int[(cos[(e_.) + (f_.)*(x_)]*(g_.))^(p_)*((a_) + (b_.)*sin[(e_.) + (f_.)*(x
_)])^(m_), x_Symbol] :> Simp[b*(g*Cos[e + f*x])^(p + 1)*((a + b*Sin[e + f*x
])^(m - 1)/(f*g*(m - 1))), x] /; FreeQ[{a, b, e, f, g, m, p}, x] && EqQ[a^2
 - b^2, 0] && EqQ[2*m + p - 1, 0] && NeQ[m, 1]
```

rule 3215
```
Int[((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_.)*((c_) + (d_.)*sin[(e_.) +
(f_.)*(x_)])^(n_.), x_Symbol] :> Simp[a^m*c^m   Int[Cos[e + f*x]^(2*m)*(c +
 d*Sin[e + f*x])^(n - m), x], x] /; FreeQ[{a, b, c, d, e, f, n}, x] && EqQ[
b*c + a*d, 0] && EqQ[a^2 - b^2, 0] && IntegerQ[m] &&  !(IntegerQ[n] && ((Lt
Q[m, 0] && GtQ[n, 0]) || LtQ[0, n, m] || LtQ[m, n, 0]))
```

rule 3252
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/((c_.) + (d_.)*sin[(e_.) + (
f_.)*(x_)]), x_Symbol] :> Simp[-2*(b/f)   Subst[Int[1/(b*c + a*d - d*x^2),
x], x, b*(Cos[e + f*x]/Sqrt[a + b*Sin[e + f*x]])], x] /; FreeQ[{a, b, c, d,
 e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]
```

rule 3413
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(sin[(e_.) + (f_.)*(x_)]*((c
_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[1/c   Int[Sqrt[a +
b*Sin[e + f*x]]/Sin[e + f*x], x], x] - Simp[d/c   Int[Sqrt[a + b*Sin[e + f*
x]]/(c + d*Sin[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c
 - a*d, 0] && EqQ[a^2 - b^2, 0]
```

3.13.4 Maple [A] (verified)

Time = 0.51 (sec) , antiderivative size = 78, normalized size of antiderivative = 1.13

method	result	size
default	$\frac{2(1+\sin(fx+e))\left(a^{\frac{3}{2}}-\operatorname{arctanh}\left(\frac{\sqrt{a-a\sin(fx+e)}}{\sqrt{a}}\right)a\sqrt{a-a\sin(fx+e)}\right)}{\sqrt{a}\,c\cos(fx+e)\sqrt{a+a\sin(fx+e)}\,f}$	78

input
```
int((a+a*sin(f*x+e))^(1/2)/sin(f*x+e)/(c-c*sin(f*x+e)),x,method=_RETURNVER
BOSE)
```

output
```
2*(1+sin(f*x+e))*(a^(3/2)-arctanh((a-a*sin(f*x+e))^(1/2)/a^(1/2))*a*(a-a*s
in(f*x+e))^(1/2))/a^(1/2)/c/cos(f*x+e)/(a+a*sin(f*x+e))^(1/2)/f
```

3.13.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 202 vs. 2(61) = 122.

Time = 0.28 (sec) , antiderivative size = 202, normalized size of antiderivative = 2.93

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{c-c\sin(e+fx)}\,dx$$
$$= \frac{\sqrt{a}\cos(fx+e)\log\left(\frac{a\cos(fx+e)^3-7a\cos(fx+e)^2-4\left(\cos(fx+e)^2+(\cos(fx+e)+3)\sin(fx+e)-2\cos(fx+e)-3\right)\sqrt{a\sin(fx+e)+a}\sqrt{a-}}{\cos(fx+e)^3+\cos(fx+e)^2+\left(\cos(fx+e)^2-1\right)\sin(fx+e)-\cos}\right.}{2cf\cos(fx+e)}$$

input
```
integrate((a+a*sin(f*x+e))^(1/2)/sin(f*x+e)/(c-c*sin(f*x+e)),x, algorithm=
"fricas")
```

output
```
1/2*(sqrt(a)*cos(f*x + e)*log((a*cos(f*x + e)^3 - 7*a*cos(f*x + e)^2 - 4*(
cos(f*x + e)^2 + (cos(f*x + e) + 3)*sin(f*x + e) - 2*cos(f*x + e) - 3)*sqr
t(a*sin(f*x + e) + a)*sqrt(a) - 9*a*cos(f*x + e) + (a*cos(f*x + e)^2 + 8*a
*cos(f*x + e) - a)*sin(f*x + e) - a)/(cos(f*x + e)^3 + cos(f*x + e)^2 + (c
os(f*x + e)^2 - 1)*sin(f*x + e) - cos(f*x + e) - 1)) + 4*sqrt(a*sin(f*x +
e) + a))/(c*f*cos(f*x + e))
```

3.13.6 Sympy [F]

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{c-c\sin(e+fx)}\,dx = -\frac{\int \frac{\sqrt{a\sin(e+fx)+a}}{\sin^2(e+fx)-\sin(e+fx)}\,dx}{c}$$

input
```
integrate((a+a*sin(f*x+e))**(1/2)/sin(f*x+e)/(c-c*sin(f*x+e)),x)
```

output
```
-Integral(sqrt(a*sin(e + f*x) + a)/(sin(e + f*x)**2 - sin(e + f*x)), x)/c
```

3.13.7 Maxima [F]

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{c-c\sin(e+fx)}\,dx = \int -\frac{\sqrt{a\sin(fx+e)+a}}{(c\sin(fx+e)-c)\sin(fx+e)}\,dx$$

input
```
integrate((a+a*sin(f*x+e))^(1/2)/sin(f*x+e)/(c-c*sin(f*x+e)),x, algorithm=
"maxima")
```

output
```
-integrate(sqrt(a*sin(f*x + e) + a)/((c*sin(f*x + e) - c)*sin(f*x + e)), x
)
```

3.13.8 Giac [A] (verification not implemented)

Time = 0.34 (sec) , antiderivative size = 110, normalized size of antiderivative = 1.59

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{c-c\sin(e+fx)}\,dx = -\frac{\sqrt{2}\left(\frac{\sqrt{2}\log\left(\frac{\left|-2\sqrt{2}+4\sin\left(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e\right)\right|}{\left|2\sqrt{2}+4\sin\left(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e\right)\right|}\right)\operatorname{sgn}\left(\cos\left(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e\right)\right)}{c}+\frac{2\operatorname{sgn}\left(\cos\left(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e\right)\right)}{c\sin\left(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e\right)}\right)\sqrt{a}}{2f}$$

input
```
integrate((a+a*sin(f*x+e))^(1/2)/sin(f*x+e)/(c-c*sin(f*x+e)),x, algorithm=
"giac")
```

output
```
-1/2*sqrt(2)*(sqrt(2)*log(abs(-2*sqrt(2) + 4*sin(-1/4*pi + 1/2*f*x + 1/2*e
))/abs(2*sqrt(2) + 4*sin(-1/4*pi + 1/2*f*x + 1/2*e)))*sgn(cos(-1/4*pi + 1/
2*f*x + 1/2*e))/c + 2*sgn(cos(-1/4*pi + 1/2*f*x + 1/2*e))/(c*sin(-1/4*pi +
 1/2*f*x + 1/2*e)))*sqrt(a)/f
```

3.13.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{c-c\sin(e+fx)}\,dx = \int \frac{\sqrt{a+a\sin(e+fx)}}{\sin(e+fx)\,(c-c\sin(e+fx))}\,dx$$

input
```
int((a + a*sin(e + f*x))^(1/2)/(sin(e + f*x)*(c - c*sin(e + f*x))),x)
```

output
```
int((a + a*sin(e + f*x))^(1/2)/(sin(e + f*x)*(c - c*sin(e + f*x))), x)
```

3.14 $\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}(c-c\sin(e+fx))}\,dx$

3.14.1 Optimal result

Integrand size = 34, antiderivative size = 120

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}(c-c\sin(e+fx))}\,dx = -\frac{2\text{arctanh}\left(\frac{\sqrt{a}\cos(e+fx)}{\sqrt{a+a\sin(e+fx)}}\right)}{\sqrt{a}cf} + \frac{\text{arctanh}\left(\frac{\sqrt{a}\cos(e+fx)}{\sqrt{2}\sqrt{a+a\sin(e+fx)}}\right)}{\sqrt{2}\sqrt{a}cf} + \frac{\sec(e+fx)\sqrt{a+a\sin(e+fx)}}{acf}$$

output
```
-2*arctanh(cos(f*x+e)*a^(1/2)/(a+a*sin(f*x+e))^(1/2))/c/f/a^(1/2)+1/2*arct
anh(1/2*cos(f*x+e)*a^(1/2)*2^(1/2)/(a+a*sin(f*x+e))^(1/2))/c/f*2^(1/2)/a^(
1/2)+sec(f*x+e)*(a+a*sin(f*x+e))^(1/2)/a/c/f
```

3.14.2 Mathematica [C] (verified)

Result contains higher order function than in optimal. Order 5 vs. order 3 in optimal.

Time = 0.32 (sec) , antiderivative size = 74, normalized size of antiderivative = 0.62

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}(c-c\sin(e+fx))}\,dx = -\frac{\left(\text{Hypergeometric2F1}\left(-\frac{1}{2},1,\frac{1}{2},\frac{1}{2}(1-\sin(e+fx))\right)-2\,\text{Hypergeometric2F1}\left(-\frac{1}{2},1,\frac{1}{2},1-\sin(e+fx)\right.\right.}{acf}$$

input
```
Integrate[Csc[e + f*x]/(Sqrt[a + a*Sin[e + f*x]]*(c - c*Sin[e + f*x])),x]
```

output
```
-(((Hypergeometric2F1[-1/2, 1, 1/2, (1 - Sin[e + f*x])/2] - 2*Hypergeometr
ic2F1[-1/2, 1, 1/2, 1 - Sin[e + f*x]])*Sec[e + f*x]*Sqrt[a*(1 + Sin[e + f*
x])])/(a*c*f))
```

3.14.3 Rubi [A] (verified)

Time = 1.09 (sec) , antiderivative size = 127, normalized size of antiderivative = 1.06, number of steps used = 13, number of rules used = 12, $\frac{\text{number of rules}}{\text{integrand size}} = 0.353$, Rules used = {3042, 3419, 3042, 3215, 3042, 3152, 3464, 3042, 3128, 219, 3252, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(e+fx)}{\sqrt{a\sin(e+fx)+a}(c-c\sin(e+fx))}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{1}{\sin(e+fx)\sqrt{a\sin(e+fx)+a}(c-c\sin(e+fx))}dx$$

$$\Big\downarrow \text{3419}$$

$$\frac{\int \frac{\csc(e+fx)(2ac+a\sin(e+fx)c)}{\sqrt{\sin(e+fx)a+a}}dx}{2ac^2}+\frac{\int \frac{\sqrt{\sin(e+fx)a+a}}{c-c\sin(e+fx)}dx}{2a}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{2ac+a\sin(e+fx)c}{\sin(e+fx)\sqrt{\sin(e+fx)a+a}}dx}{2ac^2}+\frac{\int \frac{\sqrt{\sin(e+fx)a+a}}{c-c\sin(e+fx)}dx}{2a}$$

$$\Big\downarrow \text{3215}$$

$$\frac{\int \sec^2(e+fx)(\sin(e+fx)a+a)^{3/2}dx}{2a^2c}+\frac{\int \frac{2ac+a\sin(e+fx)c}{\sin(e+fx)\sqrt{\sin(e+fx)a+a}}dx}{2ac^2}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{(\sin(e+fx)a+a)^{3/2}}{\cos(e+fx)^2}dx}{2a^2c}+\frac{\int \frac{2ac+a\sin(e+fx)c}{\sin(e+fx)\sqrt{\sin(e+fx)a+a}}dx}{2ac^2}$$

$$\Big\downarrow \text{3152}$$

$$\frac{\int \frac{2ac+a\sin(e+fx)c}{\sin(e+fx)\sqrt{\sin(e+fx)a+a}}dx}{2ac^2}+\frac{\sec(e+fx)\sqrt{a\sin(e+fx)+a}}{acf}$$

$$\Big\downarrow \text{3464}$$

$$\frac{2c\int \csc(e+fx)\sqrt{\sin(e+fx)a+a}dx-ac\int \frac{1}{\sqrt{\sin(e+fx)a+a}}dx}{2ac^2}+\frac{\sec(e+fx)\sqrt{a\sin(e+fx)+a}}{acf}$$

$$\Big\downarrow \text{3042}$$

$$\frac{2c\int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)}dx-ac\int \frac{1}{\sqrt{\sin(e+fx)a+a}}dx}{2ac^2}+\frac{\sec(e+fx)\sqrt{a\sin(e+fx)+a}}{acf}$$

$$\Big\downarrow \text{3128}$$

$$\frac{\frac{2ac\int \frac{1}{2a-\frac{a^2\cos^2(e+fx)}{\sin(e+fx)a+a}}d\frac{a\cos(e+fx)}{\sqrt{\sin(e+fx)a+a}}}{f}+2c\int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)}dx}{2ac^2}+\frac{\sec(e+fx)\sqrt{a\sin(e+fx)+a}}{acf}$$

$$\Big\downarrow \text{219}$$

$$\frac{2c\int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)}dx+\frac{\sqrt{2}\sqrt{a}c\,\text{arctanh}\left(\frac{\sqrt{a}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}}\right)}{f}}{2ac^2}+\frac{\sec(e+fx)\sqrt{a\sin(e+fx)+a}}{acf}$$

$$\Big\downarrow \text{3252}$$

$$\frac{\frac{\sqrt{2}\sqrt{a}c\,\text{arctanh}\left(\frac{\sqrt{a}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}}\right)}{f}-\frac{4ac\int \frac{1}{a-\frac{a^2\cos^2(e+fx)}{\sin(e+fx)a+a}}d\frac{a\cos(e+fx)}{\sqrt{\sin(e+fx)a+a}}}{f}}{2ac^2}+\frac{\sec(e+fx)\sqrt{a\sin(e+fx)+a}}{acf}$$

$$\Big\downarrow \text{219}$$

$$\frac{\frac{\sqrt{2}\sqrt{a}c\,\text{arctanh}\left(\frac{\sqrt{a}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}}\right)}{f}-\frac{4\sqrt{a}c\,\text{arctanh}\left(\frac{\sqrt{a}\cos(e+fx)}{\sqrt{a\sin(e+fx)+a}}\right)}{f}}{2ac^2}+\frac{\sec(e+fx)\sqrt{a\sin(e+fx)+a}}{acf}$$

input
```
Int[Csc[e + f*x]/(Sqrt[a + a*Sin[e + f*x]]*(c - c*Sin[e + f*x])),x]
```

output
```
((-4*Sqrt[a]*c*ArcTanh[(Sqrt[a]*Cos[e + f*x])/Sqrt[a + a*Sin[e + f*x]]])/f
 + (Sqrt[2]*Sqrt[a]*c*ArcTanh[(Sqrt[a]*Cos[e + f*x])/(Sqrt[2]*Sqrt[a + a*S
in[e + f*x]])])/f)/(2*a*c^2) + (Sec[e + f*x]*Sqrt[a + a*Sin[e + f*x]])/(a*
c*f)
```

3.14. $\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}(c-c\sin(e+fx))}\,dx$

3.14.3.1 Defintions of rubi rules used

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3128
```
Int[1/Sqrt[(a_) + (b_.)*sin[(c_.) + (d_.)*(x_)]], x_Symbol] :> Simp[-2/d
Subst[Int[1/(2*a - x^2), x], x, b*(Cos[c + d*x]/Sqrt[a + b*Sin[c + d*x]])],
 x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0]
```

rule 3152
```
Int[(cos[(e_.) + (f_.)*(x_)]*(g_.))^(p_)*((a_) + (b_.)*sin[(e_.) + (f_.)*(x
_)])^(m_), x_Symbol] :> Simp[b*(g*Cos[e + f*x])^(p + 1)*((a + b*Sin[e + f*x
])^(m - 1)/(f*g*(m - 1))), x] /; FreeQ[{a, b, e, f, g, m, p}, x] && EqQ[a^2
 - b^2, 0] && EqQ[2*m + p - 1, 0] && NeQ[m, 1]
```

rule 3215
```
Int[((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_.)*((c_) + (d_.)*sin[(e_.) +
(f_.)*(x_)])^(n_.), x_Symbol] :> Simp[a^m*c^m   Int[Cos[e + f*x]^(2*m)*(c +
 d*Sin[e + f*x])^(n - m), x], x] /; FreeQ[{a, b, c, d, e, f, n}, x] && EqQ[
b*c + a*d, 0] && EqQ[a^2 - b^2, 0] && IntegerQ[m] &&  !(IntegerQ[n] && ((Lt
Q[m, 0] && GtQ[n, 0]) || LtQ[0, n, m] || LtQ[m, n, 0]))
```

rule 3252
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/((c_.) + (d_.)*sin[(e_.) + (
f_.)*(x_)]), x_Symbol] :> Simp[-2*(b/f)   Subst[Int[1/(b*c + a*d - d*x^2),
x], x, b*(Cos[e + f*x]/Sqrt[a + b*Sin[e + f*x]])], x] /; FreeQ[{a, b, c, d,
 e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]
```

rule 3419
```
Int[1/(sin[(e_.) + (f_.)*(x_)]*Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]*(
(c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[d^2/(c*(b*c - a*d
))   Int[Sqrt[a + b*Sin[e + f*x]]/(c + d*Sin[e + f*x]), x], x] + Simp[1/(c*
(b*c - a*d))   Int[(b*c - a*d - b*d*Sin[e + f*x])/(Sin[e + f*x]*Sqrt[a + b*
Sin[e + f*x]]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0]
 && EqQ[a^2 - b^2, 0]
```

rule 3464
```
Int[((A_.) + (B_.)*sin[(e_.) + (f_.)*(x_)])/(Sqrt[(a_) + (b_.)*sin[(e_.) +
(f_.)*(x_)]]*((c_.) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[(A
*b - a*B)/(b*c - a*d)   Int[1/Sqrt[a + b*Sin[e + f*x]], x], x] + Simp[(B*c
- A*d)/(b*c - a*d)   Int[Sqrt[a + b*Sin[e + f*x]]/(c + d*Sin[e + f*x]), x],
 x] /; FreeQ[{a, b, c, d, e, f, A, B}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 -
 b^2, 0] && NeQ[c^2 - d^2, 0]
```

3.14.4 Maple [A] (verified)

Time = 0.39 (sec) , antiderivative size = 124, normalized size of antiderivative = 1.03

method	result
default	$\frac{(1+\sin(fx+e))\left(\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{a-a\sin(fx+e)}\,\sqrt{2}}{2\sqrt{a}}\right)a^{2}\sqrt{a-a\sin(fx+e)}+2a^{\frac{5}{2}}-4\,\operatorname{arctanh}\left(\frac{\sqrt{a-a\sin(fx+e)}}{\sqrt{a}}\right)a^{2}\sqrt{a-a\sin(fx+e)}\right)}{2c\,a^{\frac{5}{2}}\cos(fx+e)\sqrt{a+a\sin(fx+e)}\,f}$

input
```
int(1/sin(f*x+e)/(c-c*sin(f*x+e))/(a+a*sin(f*x+e))^(1/2),x,method=_RETURNV
ERBOSE)
```

output
```
1/2*(1+sin(f*x+e))*(2^(1/2)*arctanh(1/2*(a-a*sin(f*x+e))^(1/2)*2^(1/2)/a^(
1/2))*a^2*(a-a*sin(f*x+e))^(1/2)+2*a^(5/2)-4*arctanh((a-a*sin(f*x+e))^(1/2
)/a^(1/2))*a^2*(a-a*sin(f*x+e))^(1/2))/c/a^(5/2)/cos(f*x+e)/(a+a*sin(f*x+e
))^(1/2)/f
```

3.14.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 328 vs. 2(104) = 208.

Time = 0.29 (sec) , antiderivative size = 328, normalized size of antiderivative = 2.73

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}(c-c\sin(e+fx))}\,dx$$
$$= \frac{\sqrt{2}\sqrt{a}\cos(fx+e)\log\left(-\frac{\cos(fx+e)^{2}-(\cos(fx+e)-2)\sin(fx+e)+\frac{2\sqrt{2}\sqrt{a\sin(fx+e)+a}(\cos(fx+e)-\sin(fx+e)+1)}{\sqrt{a}}+3\cos(fx+e)+2}{\cos(fx+e)^{2}-(\cos(fx+e)+2)\sin(fx+e)-\cos(fx+e)-2}\right)+}{}$$

input
```
integrate(1/sin(f*x+e)/(c-c*sin(f*x+e))/(a+a*sin(f*x+e))^(1/2),x, algorith
m="fricas")
```

output
```
1/4*(sqrt(2)*sqrt(a)*cos(f*x + e)*log(-(cos(f*x + e)^2 - (cos(f*x + e) - 2
)*sin(f*x + e) + 2*sqrt(2)*sqrt(a*sin(f*x + e) + a)*(cos(f*x + e) - sin(f*
x + e) + 1)/sqrt(a) + 3*cos(f*x + e) + 2)/(cos(f*x + e)^2 - (cos(f*x + e)
+ 2)*sin(f*x + e) - cos(f*x + e) - 2)) + 2*sqrt(a)*cos(f*x + e)*log((a*cos
(f*x + e)^3 - 7*a*cos(f*x + e)^2 - 4*(cos(f*x + e)^2 + (cos(f*x + e) + 3)*
sin(f*x + e) - 2*cos(f*x + e) - 3)*sqrt(a*sin(f*x + e) + a)*sqrt(a) - 9*a*
cos(f*x + e) + (a*cos(f*x + e)^2 + 8*a*cos(f*x + e) - a)*sin(f*x + e) - a)
/(cos(f*x + e)^3 + cos(f*x + e)^2 + (cos(f*x + e)^2 - 1)*sin(f*x + e) - co
s(f*x + e) - 1)) + 4*sqrt(a*sin(f*x + e) + a))/(a*c*f*cos(f*x + e))
```

3.14.6 Sympy [F]

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}(c-c\sin(e+fx))}\,dx = -\frac{\int \frac{1}{\sqrt{a\sin{(e+fx)}+a}\sin^2{(e+fx)}-\sqrt{a\sin{(e+fx)}+a}\sin{(e+fx)}}\,dx}{c}$$

input
```
integrate(1/sin(f*x+e)/(c-c*sin(f*x+e))/(a+a*sin(f*x+e))**(1/2),x)
```

output
```
-Integral(1/(sqrt(a*sin(e + f*x) + a)*sin(e + f*x)**2 - sqrt(a*sin(e + f*x
) + a)*sin(e + f*x)), x)/c
```

3.14.7 Maxima [F]

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}(c-c\sin(e+fx))}\,dx = \int -\frac{1}{\sqrt{a\sin{(fx+e)}+a}(c\sin{(fx+e)}-c)\sin{(fx+e)}}\,dx$$

input
```
integrate(1/sin(f*x+e)/(c-c*sin(f*x+e))/(a+a*sin(f*x+e))^(1/2),x, algorith
m="maxima")
```

output
```
-integrate(1/(sqrt(a*sin(f*x + e) + a)*(c*sin(f*x + e) - c)*sin(f*x + e)),
 x)
```

3.14.8 Giac [A] (verification not implemented)

Time = 0.35 (sec) , antiderivative size = 141, normalized size of antiderivative = 1.18

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}(c-c\sin(e+fx))}\,dx =$$

$$-\frac{\sqrt{2}\left(\frac{2\sqrt{2}\log\left(\frac{\left|-2\sqrt{2}+4\sin\left(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e\right)\right|}{\left|2\sqrt{2}+4\sin\left(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e\right)\right|}\right)}{c}+\frac{\log(\sin(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e)+1)}{c}-\frac{\log(-\sin(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e)+1)}{c}+\frac{2}{c\sin(-\frac{1}{4}\pi+\frac{1}{2}}\right.}{4\sqrt{a}f\mathrm{sgn}\left(\cos\left(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e\right)\right)}$$

input
```
integrate(1/sin(f*x+e)/(c-c*sin(f*x+e))/(a+a*sin(f*x+e))^(1/2),x, algorith
m="giac")
```

output
```
-1/4*sqrt(2)*(2*sqrt(2)*log(abs(-2*sqrt(2) + 4*sin(-1/4*pi + 1/2*f*x + 1/2
*e))/abs(2*sqrt(2) + 4*sin(-1/4*pi + 1/2*f*x + 1/2*e)))/c + log(sin(-1/4*p
i + 1/2*f*x + 1/2*e) + 1)/c - log(-sin(-1/4*pi + 1/2*f*x + 1/2*e) + 1)/c +
 2/(c*sin(-1/4*pi + 1/2*f*x + 1/2*e)))/(sqrt(a)*f*sgn(cos(-1/4*pi + 1/2*f*
x + 1/2*e)))
```

3.14.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}(c-c\sin(e+fx))}\,dx = \int \frac{1}{\sin(e+f\,x)\,\sqrt{a+a\,\sin(e+f\,x)}\,(c-c\,\sin(e+f\,x))}\,dx$$

input
```
int(1/(sin(e + f*x)*(a + a*sin(e + f*x))^(1/2)*(c - c*sin(e + f*x))),x)
```

output
```
int(1/(sin(e + f*x)*(a + a*sin(e + f*x))^(1/2)*(c - c*sin(e + f*x))), x)
```

3.15 $\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}{c-c\sin(e+fx)}\,dx$

3.15.1 Optimal result

Integrand size = 40, antiderivative size = 103

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}{c-c\sin(e+fx)}\,dx = \frac{2\sqrt{a}\sqrt{g}\arctan\left(\frac{\sqrt{a}\sqrt{g}\cos(e+fx)}{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}\right)}{cf} + \frac{2\sec(e+fx)\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}{cf}$$

output
```
2*arctan(cos(f*x+e)*a^(1/2)*g^(1/2)/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^
(1/2))*a^(1/2)*g^(1/2)/c/f+2*sec(f*x+e)*(g*sin(f*x+e))^(1/2)*(a+a*sin(f*x+
e))^(1/2)/c/f
```

3.15.2 Mathematica [A] (verified)

Time = 2.59 (sec) , antiderivative size = 91, normalized size of antiderivative = 0.88

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}{c-c\sin(e+fx)}\,dx = \frac{2\sec(e+fx)\left(\arcsin\left(\sqrt{1-\sin(e+fx)}\right)\sqrt{1-\sin(e+fx)}+\sqrt{\sin(e+fx)}\right)\sqrt{g\sin(e+fx)}\sqrt{a(1+\sin}}{cf\sqrt{\sin(e+fx)}}$$

input
```
Integrate[(Sqrt[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]])/(c - c*Sin[e + f
*x]),x]
```

output
```
(2*Sec[e + f*x]*(ArcSin[Sqrt[1 - Sin[e + f*x]]]*Sqrt[1 - Sin[e + f*x]] + S
qrt[Sin[e + f*x]])*Sqrt[g*Sin[e + f*x]]*Sqrt[a*(1 + Sin[e + f*x])])/(c*f*S
qrt[Sin[e + f*x]])
```

3.15.3 Rubi [A] (verified)

Time = 0.85 (sec) , antiderivative size = 103, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.175, Rules used = {3042, 3407, 3042, 3254, 218, 3409, 15}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}{c-c\sin(e+fx)}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}{c-c\sin(e+fx)}dx$$

$$\Big\downarrow \text{3407}$$

$$g\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}(c-c\sin(e+fx))}dx-\frac{g\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}}dx}{c}$$

$$\Big\downarrow \text{3042}$$

$$g\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}(c-c\sin(e+fx))}dx-\frac{g\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}}dx}{c}$$

$$\Big\downarrow \text{3254}$$

$$\frac{2ag\int \frac{1}{\frac{\cos(e+fx)\cot(e+fx)a^2}{\sin(e+fx)a+a}+a}d\frac{a\cos(e+fx)}{\sqrt{g\sin(e+fx)}\sqrt{\sin(e+fx)a+a}}}{cf}+g\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}(c-c\sin(e+fx))}dx$$

$$\Big\downarrow \text{218}$$

$$g\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}(c-c\sin(e+fx))}dx+\frac{2\sqrt{a}\sqrt{g}\arctan\left(\frac{\sqrt{a}\sqrt{g}\cos(e+fx)}{\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}\right)}{cf}$$

$$\Big\downarrow \text{3409}$$

$$\frac{2\sqrt{a}\sqrt{g}\arctan\left(\frac{\sqrt{a}\sqrt{g}\cos(e+fx)}{\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}\right)}{cf}-\frac{2ag\int\frac{\sec(e+fx)(\sin(e+fx)a+a)\tan(e+fx)}{a^2c}d\frac{a\cos(e+fx)}{\sqrt{g\sin(e+fx)}\sqrt{\sin(e+fx)a+a}}}{f}$$

$$\Big\downarrow \text{15}$$

$$\frac{2\sqrt{a}\sqrt{g}\arctan\left(\frac{\sqrt{a}\sqrt{g}\cos(e+fx)}{\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}\right)}{cf}+\frac{2\sec(e+fx)\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}{cf}$$

input
```
Int[(Sqrt[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]])/(c - c*Sin[e + f*x]),x
]
```

output
```
(2*Sqrt[a]*Sqrt[g]*ArcTan[(Sqrt[a]*Sqrt[g]*Cos[e + f*x])/(Sqrt[g*Sin[e + f
*x]]*Sqrt[a + a*Sin[e + f*x]])])/(c*f) + (2*Sec[e + f*x]*Sqrt[g*Sin[e + f*
x]]*Sqrt[a + a*Sin[e + f*x]])/(c*f)
```

3.15.3.1 Defintions of rubi rules used

rule 15
```
Int[(a_.)*(x_)^(m_.), x_Symbol] :> Simp[a*(x^(m + 1)/(m + 1)), x] /; FreeQ[
{a, m}, x] && NeQ[m, -1]
```

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3254
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/Sqrt[(c_.) + (d_.)*sin[(e_.)
 + (f_.)*(x_)]], x_Symbol] :> Simp[-2*(b/f)   Subst[Int[1/(b + d*x^2), x],
x, b*(Cos[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]]))], x
] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0]
 && NeQ[c^2 - d^2, 0]
```

rule 3407
```
Int[(Sqrt[(g_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.
)*(x_)]])/((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] :> Simp[g/d   I
nt[Sqrt[a + b*Sin[e + f*x]]/Sqrt[g*Sin[e + f*x]], x], x] - Simp[c*(g/d)   I
nt[Sqrt[a + b*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c + d*Sin[e + f*x])), x]
, x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && (EqQ[a^2 -
b^2, 0] || EqQ[c^2 - d^2, 0])
```

rule 3409
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(g_.)*sin[(e_.) + (f_.
)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[-2*(b/f
)   Subst[Int[1/(b*c + a*d + c*g*x^2), x], x, b*(Cos[e + f*x]/(Sqrt[g*Sin[e
 + f*x]]*Sqrt[a + b*Sin[e + f*x]]))], x] /; FreeQ[{a, b, c, d, e, f, g}, x]
 && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0]
```

3.15.4 Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 753 vs. $2(87) = 174$.

Time = 3.21 (sec) , antiderivative size = 754, normalized size of antiderivative = 7.32

method	result
default	$-\frac{\sqrt{\frac{g(\csc(fx+e)-\cot(fx+e))}{(1-\cos(fx+e))^2\left(\csc^2(fx+e)\right)+1}}\left((1-\cos(fx+e))^2\left(\csc^2(fx+e)\right)+1\right)\sqrt{\frac{a\left((1-\cos(fx+e))^2\left(\csc^2(fx+e)\right)+2\csc(fx+e)-2\cot(fx+e)+1\right)}{(1-\cos(fx+e))^2\left(\csc^2(fx+e)\right)+1}}}{}$

input
```
int((g*sin(f*x+e))^(1/2)*(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x,method=
_RETURNVERBOSE)
```

output
```
-1/4/c/f*(g/((1-cos(f*x+e))^2*csc(f*x+e)^2+1)*(csc(f*x+e)-cot(f*x+e)))^(1/
2)*((1-cos(f*x+e))^2*csc(f*x+e)^2+1)*(a*((1-cos(f*x+e))^2*csc(f*x+e)^2+2*c
sc(f*x+e)-2*cot(f*x+e)+1)/((1-cos(f*x+e))^2*csc(f*x+e)^2+1))^(1/2)/(-cot(f
*x+e)+csc(f*x+e)+1)*(2^(1/2)*ln(-(csc(f*x+e)-cot(f*x+e)+(csc(f*x+e)-cot(f*
x+e))^(1/2)*2^(1/2)+1)/((csc(f*x+e)-cot(f*x+e))^(1/2)*2^(1/2)-csc(f*x+e)+c
ot(f*x+e)-1))*(csc(f*x+e)-cot(f*x+e))+4*2^(1/2)*arctan((csc(f*x+e)-cot(f*x
+e))^(1/2)*2^(1/2)+1)*(csc(f*x+e)-cot(f*x+e))+4*2^(1/2)*arctan((csc(f*x+e)
-cot(f*x+e))^(1/2)*2^(1/2)-1)*(csc(f*x+e)-cot(f*x+e))+2^(1/2)*ln(-((csc(f*
x+e)-cot(f*x+e))^(1/2)*2^(1/2)-csc(f*x+e)+cot(f*x+e)-1)/(csc(f*x+e)-cot(f*
x+e)+(csc(f*x+e)-cot(f*x+e))^(1/2)*2^(1/2)+1))*(csc(f*x+e)-cot(f*x+e))-2^(
1/2)*ln(-(csc(f*x+e)-cot(f*x+e)+(csc(f*x+e)-cot(f*x+e))^(1/2)*2^(1/2)+1)/(
(csc(f*x+e)-cot(f*x+e))^(1/2)*2^(1/2)-csc(f*x+e)+cot(f*x+e)-1))-4*2^(1/2)*
arctan((csc(f*x+e)-cot(f*x+e))^(1/2)*2^(1/2)+1)-4*2^(1/2)*arctan((csc(f*x+
e)-cot(f*x+e))^(1/2)*2^(1/2)-1)-2^(1/2)*ln(-((csc(f*x+e)-cot(f*x+e))^(1/2)
*2^(1/2)-csc(f*x+e)+cot(f*x+e)-1)/(csc(f*x+e)-cot(f*x+e)+(csc(f*x+e)-cot(f
*x+e))^(1/2)*2^(1/2)+1))+8*(csc(f*x+e)-cot(f*x+e))^(1/2))/(csc(f*x+e)-cot(
f*x+e))^(1/2)/((csc(f*x+e)-cot(f*x+e))^(1/2)+1)/((csc(f*x+e)-cot(f*x+e))^(
1/2)-1)*2^(1/2)
```

3.15.5 Fricas [A] (verification not implemented)

Time = 0.41 (sec) , antiderivative size = 442, normalized size of antiderivative = 4.29

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}{c-c\sin(e+fx)}\,dx$$

$$= \left[\frac{\sqrt{-ag}\cos\left(fx+e\right)\log\left(\frac{128\,ag\cos(fx+e)^5-128\,ag\cos(fx+e)^4-416\,ag\cos(fx+e)^3+128\,ag\cos(fx+e)^2+289\,ag\cos(fx+e)+8\left(16\text{ c}\right.}{}\right.}{}\right.$$

$$-\frac{\sqrt{ag}\arctan\left(\frac{\sqrt{ag}\left(8\cos(fx+e)^2+8\sin(fx+e)-9\right)\sqrt{a\sin(fx+e)+a}\sqrt{g\sin(fx+e)}}{4\left(2\,ag\cos(fx+e)^3+ag\cos(fx+e)\sin(fx+e)-2\,ag\cos(fx+e)\right)}\right)\cos\left(fx+e\right)-4\sqrt{a\sin\left(fx+e\right)+}}{2\,cf\cos\left(fx+e\right)}$$

input
```
integrate((g*sin(f*x+e))^(1/2)*(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x,
algorithm="fricas")
```

output
```
[1/4*(sqrt(-a*g)*cos(f*x + e)*log((128*a*g*cos(f*x + e)^5 - 128*a*g*cos(f*
x + e)^4 - 416*a*g*cos(f*x + e)^3 + 128*a*g*cos(f*x + e)^2 + 289*a*g*cos(f
*x + e) + 8*(16*cos(f*x + e)^4 - 24*cos(f*x + e)^3 - 66*cos(f*x + e)^2 + (
16*cos(f*x + e)^3 + 40*cos(f*x + e)^2 - 26*cos(f*x + e) - 51)*sin(f*x + e)
 + 25*cos(f*x + e) + 51)*sqrt(-a*g)*sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*
x + e)) + a*g + (128*a*g*cos(f*x + e)^4 + 256*a*g*cos(f*x + e)^3 - 160*a*g
*cos(f*x + e)^2 - 288*a*g*cos(f*x + e) + a*g)*sin(f*x + e))/(cos(f*x + e)
+ sin(f*x + e) + 1)) + 8*sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*x + e)))/(c
*f*cos(f*x + e)), -1/2*(sqrt(a*g)*arctan(1/4*sqrt(a*g)*(8*cos(f*x + e)^2 +
 8*sin(f*x + e) - 9)*sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))/(2*a*g*
cos(f*x + e)^3 + a*g*cos(f*x + e)*sin(f*x + e) - 2*a*g*cos(f*x + e)))*cos(
f*x + e) - 4*sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*x + e)))/(c*f*cos(f*x +
 e))]
```

3.15.6 Sympy [F]

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}{c-c\sin(e+fx)}\,dx = -\frac{\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a\sin(e+fx)+a}}{\sin(e+fx)-1}\,dx}{c}$$

input
```
integrate((g*sin(f*x+e))**(1/2)*(a+a*sin(f*x+e))**(1/2)/(c-c*sin(f*x+e)),x
)
```

output
```
-Integral(sqrt(g*sin(e + f*x))*sqrt(a*sin(e + f*x) + a)/(sin(e + f*x) - 1)
, x)/c
```

3.15.7 Maxima [F]

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}{c-c\sin(e+fx)}\,dx = \int -\frac{\sqrt{a\sin(fx+e)+a}\sqrt{g\sin(fx+e)}}{c\sin(fx+e)-c}\,dx$$

input
```
integrate((g*sin(f*x+e))^(1/2)*(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x,
algorithm="maxima")
```

output
```
-integrate(sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))/(c*sin(f*x + e) -
 c), x)
```

3.15.8 Giac [F(-1)]

Timed out.

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}{c-c\sin(e+fx)}\,dx = \text{Timed out}$$

input
```
integrate((g*sin(f*x+e))^(1/2)*(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x,
algorithm="giac")
```

output
```
Timed out
```

3.15.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}{c-c\sin(e+fx)}\,dx = \int \frac{\sqrt{g\,\sin\left(e+f\,x\right)}\,\sqrt{a+a\,\sin\left(e+f\,x\right)}}{c-c\,\sin\left(e+f\,x\right)}\,dx$$

input
```
int(((g*sin(e + f*x))^(1/2)*(a + a*sin(e + f*x))^(1/2))/(c - c*sin(e + f*x
)),x)
```

output
```
int(((g*sin(e + f*x))^(1/2)*(a + a*sin(e + f*x))^(1/2))/(c - c*sin(e + f*x
)), x)
```

3.16 $\int \frac{\sqrt{a+a\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c-c\sin(e+fx))}\,dx$

3.16.1 Optimal result

Integrand size = 40, antiderivative size = 43

$$\int \frac{\sqrt{a+a\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c-c\sin(e+fx))}\,dx = \frac{2\sec(e+fx)\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}{cfg}$$

output
```
2*sec(f*x+e)*(g*sin(f*x+e))^(1/2)*(a+a*sin(f*x+e))^(1/2)/c/f/g
```

3.16.2 Mathematica [A] (verified)

Time = 0.55 (sec) , antiderivative size = 40, normalized size of antiderivative = 0.93

$$\int \frac{\sqrt{a+a\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c-c\sin(e+fx))}\,dx = \frac{2\sqrt{a(1+\sin(e+fx))}\tan(e+fx)}{cf\sqrt{g\sin(e+fx)}}$$

input
```
Integrate[Sqrt[a + a*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c - c*Sin[e + f*
x])),x]
```

output
```
(2*Sqrt[a*(1 + Sin[e + f*x])]*Tan[e + f*x])/(c*f*Sqrt[g*Sin[e + f*x]])
```

3.16.3 Rubi [A] (verified)

Time = 0.37 (sec) , antiderivative size = 43, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.075, Rules used = {3042, 3409, 15}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sqrt{a\sin(e+fx)+a}}{(c-c\sin(e+fx))\sqrt{g\sin(e+fx)}}dx$$
$$\Big\downarrow \ 3042$$
$$\int \frac{\sqrt{a\sin(e+fx)+a}}{(c-c\sin(e+fx))\sqrt{g\sin(e+fx)}}dx$$
$$\Big\downarrow \ 3409$$
$$-\frac{2a\int \frac{\sec(e+fx)(\sin(e+fx)a+a)\tan(e+fx)}{a^2c}d\frac{a\cos(e+fx)}{\sqrt{g\sin(e+fx)}\sqrt{\sin(e+fx)a+a}}}{f}$$
$$\Big\downarrow \ 15$$
$$\frac{2\sec(e+fx)\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}{cfg}$$

input
```
Int[Sqrt[a + a*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c - c*Sin[e + f*x])),x
]
```

output
```
(2*Sec[e + f*x]*Sqrt[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]])/(c*f*g)
```

3.16.3.1 Defintions of rubi rules used

rule 15
```
Int[(a_.)*(x_)^(m_.), x_Symbol] :> Simp[a*(x^(m + 1)/(m + 1)), x] /; FreeQ[
{a, m}, x] && NeQ[m, -1]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3409
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(g_.)*sin[(e_.) + (f_.
)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[-2*(b/f
)   Subst[Int[1/(b*c + a*d + c*g*x^2), x], x, b*(Cos[e + f*x]/(Sqrt[g*Sin[e
 + f*x]]*Sqrt[a + b*Sin[e + f*x]]))], x] /; FreeQ[{a, b, c, d, e, f, g}, x]
 && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0]
```

3.16.4 Maple [A] (verified)

Time = 2.86 (sec) , antiderivative size = 37, normalized size of antiderivative = 0.86

method	result	size
default	$\frac{2\tan(fx+e)\sqrt{a(1+\sin(fx+e))}}{cf\sqrt{g\sin(fx+e)}}$	37

input
```
int((a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))/(g*sin(f*x+e))^(1/2),x,method=
_RETURNVERBOSE)
```

output
```
2/c/f*tan(f*x+e)*(a*(1+sin(f*x+e)))^(1/2)/(g*sin(f*x+e))^(1/2)
```

3.16.5 Fricas [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 41, normalized size of antiderivative = 0.95

$$\int \frac{\sqrt{a+a\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c-c\sin(e+fx))}\,dx = \frac{2\sqrt{a\sin(fx+e)+a}\sqrt{g\sin(fx+e)}}{cfg\cos(fx+e)}$$

input
```
integrate((a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))/(g*sin(f*x+e))^(1/2),x,
algorithm="fricas")
```

output
```
2*sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))/(c*f*g*cos(f*x + e))
```

3.16.6 Sympy [F]

$$\int \frac{\sqrt{a+a\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c-c\sin(e+fx))}\,dx = -\frac{\int \frac{\sqrt{a\sin(e+fx)+a}}{\sqrt{g\sin(e+fx)}\sin(e+fx)-\sqrt{g\sin(e+fx)}}\,dx}{c}$$

input
```
integrate((a+a*sin(f*x+e))**(1/2)/(c-c*sin(f*x+e))/(g*sin(f*x+e))**(1/2),x
)
```

output
```
-Integral(sqrt(a*sin(e + f*x) + a)/(sqrt(g*sin(e + f*x))*sin(e + f*x) - sq
rt(g*sin(e + f*x))), x)/c
```

3.16.7 Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 309 vs. $2(39) = 78$.

Time = 0.31 (sec) , antiderivative size = 309, normalized size of antiderivative = 7.19

$$\int \frac{\sqrt{a+a\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c-c\sin(e+fx))}\,dx =$$
$$-\frac{\frac{4\left(\left(\frac{3\sqrt{2}\sqrt{a}\sqrt{g}\sin(fx+e)}{\cos(fx+e)+1}-\frac{\sqrt{2}\sqrt{a}\sqrt{g}\sin(fx+e)^2}{(\cos(fx+e)+1)^2}\right)\sqrt{\frac{\sin(fx+e)}{\cos(fx+e)+1}}-\frac{2\left(\frac{3\sqrt{2}\sqrt{a}\sqrt{g}\sin(fx+e)}{\cos(fx+e)+1}+\frac{\sqrt{2}\sqrt{a}\sqrt{g}\sin(fx+e)^2}{(\cos(fx+e)+1)^2}\right)}{\sqrt{\frac{\sin(fx+e)}{\cos(fx+e)+1}}}\right)}{cg-\frac{cg\sin(fx+e)}{\cos(fx+e)+1}}-\frac{2\sqrt{2}\sqrt{a}\left(\frac{\sin(fx+e)}{\cos(fx+e)+1}\right)^{\frac{3}{2}}\cdots}{c\sqrt{g}}\cdots}{12\,f}$$

input
```
integrate((a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))/(g*sin(f*x+e))^(1/2),x,
algorithm="maxima")
```

output
```
-1/12*(4*((3*sqrt(2)*sqrt(a)*sqrt(g)*sin(f*x + e)/(cos(f*x + e) + 1) - sqr
t(2)*sqrt(a)*sqrt(g)*sin(f*x + e)^2/(cos(f*x + e) + 1)^2)*sqrt(sin(f*x + e
)/(cos(f*x + e) + 1)) - 2*(3*sqrt(2)*sqrt(a)*sqrt(g)*sin(f*x + e)/(cos(f*x
 + e) + 1) + sqrt(2)*sqrt(a)*sqrt(g)*sin(f*x + e)^2/(cos(f*x + e) + 1)^2)/
sqrt(sin(f*x + e)/(cos(f*x + e) + 1)))/(c*g - c*g*sin(f*x + e)/(cos(f*x +
e) + 1)) - (2*sqrt(2)*sqrt(a)*(sin(f*x + e)/(cos(f*x + e) + 1))^(3/2) + 3*
sqrt(2)*sqrt(a)*sin(f*x + e)/(cos(f*x + e) + 1))/(c*sqrt(g)) - (2*sqrt(2)*
sqrt(a)*(sin(f*x + e)/(cos(f*x + e) + 1))^(3/2) - 3*sqrt(2)*sqrt(a)*sin(f*
x + e)/(cos(f*x + e) + 1))/(c*sqrt(g)))/f
```

3.16.8 Giac [F]

$$\int \frac{\sqrt{a+a\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c-c\sin(e+fx))}\,dx = \int -\frac{\sqrt{a\sin(fx+e)+a}}{(c\sin(fx+e)-c)\sqrt{g\sin(fx+e)}}\,dx$$

input
```
integrate((a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))/(g*sin(f*x+e))^(1/2),x,
algorithm="giac")
```

output
```
integrate(-sqrt(a*sin(f*x + e) + a)/((c*sin(f*x + e) - c)*sqrt(g*sin(f*x +
 e))), x)
```

3.16.9 Mupad [B] (verification not implemented)

Time = 13.13 (sec) , antiderivative size = 52, normalized size of antiderivative = 1.21

$$\int \frac{\sqrt{a+a\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c-c\sin(e+fx))}\,dx = \frac{2\sin(2e+2fx)\sqrt{a(\sin(e+fx)+1)}}{cf(\cos(2e+2fx)+1)\sqrt{g\sin(e+fx)}}$$

input
```
int((a + a*sin(e + f*x))^(1/2)/((g*sin(e + f*x))^(1/2)*(c - c*sin(e + f*x)
)),x)
```

output
```
(2*sin(2*e + 2*f*x)*(a*(sin(e + f*x) + 1))^(1/2))/(c*f*(cos(2*e + 2*f*x) +
 1)*(g*sin(e + f*x))^(1/2))
```

3.17 $\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}(c-c\sin(e+fx))}\,dx$

3.17.1 Optimal result

Integrand size = 40, antiderivative size = 114

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}(c-c\sin(e+fx))}\,dx = \frac{\sqrt{g}\arctan\left(\frac{\sqrt{a}\sqrt{g}\cos(e+fx)}{\sqrt{2}\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}\right)}{\sqrt{2}\sqrt{a}cf} + \frac{\sec(e+fx)\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}{acf}$$

output
```
1/2*arctan(1/2*cos(f*x+e)*a^(1/2)*g^(1/2)*2^(1/2)/(g*sin(f*x+e))^(1/2)/(a+
a*sin(f*x+e))^(1/2))*g^(1/2)/c/f*2^(1/2)/a^(1/2)+sec(f*x+e)*(g*sin(f*x+e))
^(1/2)*(a+a*sin(f*x+e))^(1/2)/a/c/f
```

3.17.2 Mathematica [A] (verified)

Time = 0.22 (sec) , antiderivative size = 133, normalized size of antiderivative = 1.17

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}(c-c\sin(e+fx))}\,dx = \frac{\csc(2(e+fx))\sqrt{\sin(e+fx)}\sqrt{g\sin(e+fx)}\sqrt{a(1+\sin(e+fx))}\left(2\sqrt{c}\sqrt{\sin(e+fx)}-\sqrt{2}\arctan\left(\frac{\sqrt{2}\sqrt{c}\cdots}{\sqrt{c-c\cdots}}\right.\right.}{ac^{3/2}f}$$

input
```
Integrate[Sqrt[g*Sin[e + f*x]]/(Sqrt[a + a*Sin[e + f*x]]*(c - c*Sin[e + f*
x])),x]
```

output
```
(Csc[2*(e + f*x)]*Sqrt[Sin[e + f*x]]*Sqrt[g*Sin[e + f*x]]*Sqrt[a*(1 + Sin[
e + f*x])]*(2*Sqrt[c]*Sqrt[Sin[e + f*x]] - Sqrt[2]*ArcTan[(Sqrt[2]*Sqrt[c]
*Sqrt[Sin[e + f*x]])/Sqrt[c - c*Sin[e + f*x]]]*Sqrt[c - c*Sin[e + f*x]]))/
(a*c^(3/2)*f)
```

3.17.3 Rubi [A] (verified)

Time = 0.86 (sec) , antiderivative size = 114, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}} = 0.175$, Rules used = {3042, 3415, 3042, 3261, 218, 3409, 15}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a\sin(e+fx)+a}(c-c\sin(e+fx))}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a\sin(e+fx)+a}(c-c\sin(e+fx))}dx$$

$$\Big\downarrow \text{3415}$$

$$\frac{g\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}(c-c\sin(e+fx))}dx}{2a}-\frac{g\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{\sin(e+fx)a+a}}dx}{2c}$$

$$\Big\downarrow \text{3042}$$

$$\frac{g\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}(c-c\sin(e+fx))}dx}{2a}-\frac{g\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{\sin(e+fx)a+a}}dx}{2c}$$

$$\Big\downarrow \text{3261}$$

$$\frac{ag\int \frac{1}{\frac{\cos(e+fx)\cot(e+fx)a^3}{\sin(e+fx)a+a}+2a^2}d\frac{a\cos(e+fx)}{\sqrt{g\sin(e+fx)}\sqrt{\sin(e+fx)a+a}}}{cf}+\frac{g\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}(c-c\sin(e+fx))}dx}{2a}$$

$$\Big\downarrow \text{218}$$

$$\frac{g\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}(c-c\sin(e+fx))}dx}{2a}+\frac{\sqrt{g}\arctan\left(\frac{\sqrt{a}\sqrt{g}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}\right)}{\sqrt{2}\sqrt{a}cf}$$

$$\Big\downarrow \text{ 3409}$$

$$\frac{\sqrt{g}\arctan\left(\frac{\sqrt{a}\sqrt{g}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}\right)}{\sqrt{2}\sqrt{a}cf}-\frac{g\int\frac{\sec(e+fx)(\sin(e+fx)a+a)\tan(e+fx)}{a^2c}d\frac{a\cos(e+fx)}{\sqrt{g\sin(e+fx)}\sqrt{\sin(e+fx)a+a}}}{f}$$

$$\Big\downarrow \text{ 15}$$

$$\frac{\sqrt{g}\arctan\left(\frac{\sqrt{a}\sqrt{g}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}\right)}{\sqrt{2}\sqrt{a}cf}+\frac{\sec(e+fx)\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}{acf}$$

input
```
Int[Sqrt[g*Sin[e + f*x]]/(Sqrt[a + a*Sin[e + f*x]]*(c - c*Sin[e + f*x])),x
]
```

output
```
(Sqrt[g]*ArcTan[(Sqrt[a]*Sqrt[g]*Cos[e + f*x])/(Sqrt[2]*Sqrt[g*Sin[e + f*x
]]*Sqrt[a + a*Sin[e + f*x]])]])/(Sqrt[2]*Sqrt[a]*c*f) + (Sec[e + f*x]*Sqrt[
g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]])/(a*c*f)
```

3.17.3.1 Defintions of rubi rules used

rule 15
```
Int[(a_.)*(x_)^(m_.), x_Symbol] :> Simp[a*(x^(m + 1)/(m + 1)), x] /; FreeQ[
{a, m}, x] && NeQ[m, -1]
```

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3261
```
Int[1/(Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(c_.) + (d_.)*sin[(e
_.) + (f_.)*(x_)]]), x_Symbol] :> Simp[-2*(a/f)   Subst[Int[1/(2*b^2 - (a*c
 - b*d)*x^2), x], x, b*(Cos[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*S
in[e + f*x]]))], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] &&
 EqQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]
```

rule 3409
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(g_.)*sin[(e_.) + (f_.
)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[-2*(b/f
)   Subst[Int[1/(b*c + a*d + c*g*x^2), x], x, b*(Cos[e + f*x]/(Sqrt[g*Sin[e
 + f*x]]*Sqrt[a + b*Sin[e + f*x]]))], x] /; FreeQ[{a, b, c, d, e, f, g}, x]
 && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0]
```

rule 3415
```
Int[Sqrt[(g_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.
)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[(-a)*(g
/(b*c - a*d))   Int[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]), x],
x] + Simp[c*(g/(b*c - a*d))   Int[Sqrt[a + b*Sin[e + f*x]]/(Sqrt[g*Sin[e +
f*x]]*(c + d*Sin[e + f*x])), x], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && N
eQ[b*c - a*d, 0] && (EqQ[a^2 - b^2, 0] || EqQ[c^2 - d^2, 0])
```

3.17.4 Maple [A] (verified)

Time = 3.34 (sec) , antiderivative size = 158, normalized size of antiderivative = 1.39

method	result
default	$-\frac{\left(\sqrt{\csc(fx+e)-\cot(fx+e)}\,\sin(fx+e)-2\arctan\left(\sqrt{\csc(fx+e)-\cot(fx+e)}\right)\cos(fx+e)+\sqrt{\csc(fx+e)-\cot(fx+e)}\,\cos(fx+e)+\sqrt{}\right.}{cf(-\cos(fx+e)+\sin(fx+e)-1)\sqrt{a(1+\sin(fx+e))}\,\sqrt{\csc(fx+e)-\cot(fx+e)}}$

input
```
int((g*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))/(a+a*sin(f*x+e))^(1/2),x,method=
_RETURNVERBOSE)
```

output
```
-1/c/f*((csc(f*x+e)-cot(f*x+e))^(1/2)*sin(f*x+e)-2*arctan((csc(f*x+e)-cot(
f*x+e))^(1/2))*cos(f*x+e)+(csc(f*x+e)-cot(f*x+e))^(1/2)*cos(f*x+e)+(csc(f*
x+e)-cot(f*x+e))^(1/2))*(g*sin(f*x+e))^(1/2)/(-cos(f*x+e)+sin(f*x+e)-1)/(a
*(1+sin(f*x+e)))^(1/2)/(csc(f*x+e)-cot(f*x+e))^(1/2)
```

3.17.5 Fricas [A] (verification not implemented)

Time = 0.36 (sec) , antiderivative size = 385, normalized size of antiderivative = 3.38

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}(c-c\sin(e+fx))}\,dx$$
$$= \left[\frac{\sqrt{2}a\sqrt{-\frac{g}{a}}\cos(fx+e)\log\left(\frac{17\,g\cos(fx+e)^3+4\sqrt{2}\left(3\cos(fx+e)^2+(3\cos(fx+e)+4)\sin(fx+e)-\cos(fx+e)-4\right)\sqrt{a\sin(fx+e)+a}\sqrt{\cdots}}{\cos(fx+e)^3+3\cos(fx+e)^2+\left(\cos(fx+e)\cdots\right.}\right.}{\cdots}\right.$$
$$-\frac{\sqrt{2}a\sqrt{\frac{g}{a}}\arctan\left(\frac{\sqrt{2}\sqrt{a\sin(fx+e)+a}\sqrt{g\sin(fx+e)}\sqrt{\frac{g}{a}}(3\sin(fx+e)-1)}{4\,g\cos(fx+e)\sin(fx+e)}\right)\cos(fx+e)-4\sqrt{a\sin(fx+e)+a}\sqrt{g\,s\cdots}}{4\,acf\cos(fx+e)}$$

input
```
integrate((g*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))/(a+a*sin(f*x+e))^(1/2),x,
algorithm="fricas")
```

output
```
[1/8*(sqrt(2)*a*sqrt(-g/a)*cos(f*x + e)*log((17*g*cos(f*x + e)^3 + 4*sqrt(
2)*(3*cos(f*x + e)^2 + (3*cos(f*x + e) + 4)*sin(f*x + e) - cos(f*x + e) -
4)*sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))*sqrt(-g/a) + 3*g*cos(f*x
+ e)^2 - 18*g*cos(f*x + e) + (17*g*cos(f*x + e)^2 + 14*g*cos(f*x + e) - 4*
g)*sin(f*x + e) - 4*g)/(cos(f*x + e)^3 + 3*cos(f*x + e)^2 + (cos(f*x + e)^
2 - 2*cos(f*x + e) - 4)*sin(f*x + e) - 2*cos(f*x + e) - 4)) + 8*sqrt(a*sin
(f*x + e) + a)*sqrt(g*sin(f*x + e)))/(a*c*f*cos(f*x + e)), -1/4*(sqrt(2)*a
*sqrt(g/a)*arctan(1/4*sqrt(2)*sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*x + e)
)*sqrt(g/a)*(3*sin(f*x + e) - 1)/(g*cos(f*x + e)*sin(f*x + e)))*cos(f*x +
e) - 4*sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*x + e)))/(a*c*f*cos(f*x + e))
]
```

3.17.6 Sympy [F]

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}(c-c\sin(e+fx))}\,dx = -\frac{\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a\sin(e+fx)+a}\sin(e+fx)-\sqrt{a\sin(e+fx)+a}}\,dx}{c}$$

input
```
integrate((g*sin(f*x+e))**(1/2)/(c-c*sin(f*x+e))/(a+a*sin(f*x+e))**(1/2),x
)
```

output
```
-Integral(sqrt(g*sin(e + f*x))/(sqrt(a*sin(e + f*x) + a)*sin(e + f*x) - sq
rt(a*sin(e + f*x) + a)), x)/c
```

3.17.7 Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 270 vs. $2(96) = 192$.

Time = 0.32 (sec) , antiderivative size = 270, normalized size of antiderivative = 2.37

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}(c-c\sin(e+fx))}\,dx = \frac{\frac{4\sqrt{2}\sqrt{g}\left(\frac{\sin(fx+e)}{\cos(fx+e)+1}\right)^{\frac{3}{2}}}{\sqrt{a}c+\frac{\sqrt{a}c\sin(fx+e)}{\cos(fx+e)+1}-\frac{\left(\sqrt{a}c+\frac{\sqrt{a}c\sin(fx+e)}{\cos(fx+e)+1}\right)\sin(fx+e)}{\cos(fx+e)+1}} - \frac{2\sqrt{2}\sqrt{g}\arctan\left(\sqrt{\frac{\sin(fx+e)}{\cos(fx+e)+1}}\right)}{\sqrt{a}c} + \frac{\sqrt{2}\sqrt{g}\sqrt{\frac{\sin(fx+e)}{\cos(fx+e)+1}}+\sqrt{2}\sqrt{g}}{\sqrt{a}c+\frac{\sqrt{a}c\sin(fx+e)}{\cos(fx+e)+1}} + \frac{\sqrt{2}\sqrt{g}\sqrt{\frac{\sin(f}{\cos(fx}}}{\sqrt{a}c+\frac{\sqrt{a}c}{\cos(}}}{2f}$$

input
```
integrate((g*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))/(a+a*sin(f*x+e))^(1/2),x,
algorithm="maxima")
```

output
```
1/2*(4*sqrt(2)*sqrt(g)*(sin(f*x + e)/(cos(f*x + e) + 1))^(3/2)/(sqrt(a)*c
+ sqrt(a)*c*sin(f*x + e)/(cos(f*x + e) + 1) - (sqrt(a)*c + sqrt(a)*c*sin(f
*x + e)/(cos(f*x + e) + 1))*sin(f*x + e)/(cos(f*x + e) + 1)) - 2*sqrt(2)*s
qrt(g)*arctan(sqrt(sin(f*x + e)/(cos(f*x + e) + 1)))/(sqrt(a)*c) + (sqrt(2
)*sqrt(g)*sqrt(sin(f*x + e)/(cos(f*x + e) + 1)) + sqrt(2)*sqrt(g))/(sqrt(a
)*c + sqrt(a)*c*sin(f*x + e)/(cos(f*x + e) + 1)) + (sqrt(2)*sqrt(g)*sqrt(s
in(f*x + e)/(cos(f*x + e) + 1)) - sqrt(2)*sqrt(g))/(sqrt(a)*c + sqrt(a)*c*
sin(f*x + e)/(cos(f*x + e) + 1)))/f
```

3.17.8 Giac [F(-1)]

Timed out.

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}(c-c\sin(e+fx))}\,dx = \text{Timed out}$$

input
```
integrate((g*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))/(a+a*sin(f*x+e))^(1/2),x,
algorithm="giac")
```

output
```
Timed out
```

3.17.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}(c-c\sin(e+fx))}\,dx$$
$$=\int \frac{\sqrt{g\,\sin\left(e+f\,x\right)}}{\sqrt{a+a\,\sin\left(e+f\,x\right)}\,\left(c-c\,\sin\left(e+f\,x\right)\right)}\,dx$$

input
```
int((g*sin(e + f*x))^(1/2)/((a + a*sin(e + f*x))^(1/2)*(c - c*sin(e + f*x)
)),x)
```

output
```
int((g*sin(e + f*x))^(1/2)/((a + a*sin(e + f*x))^(1/2)*(c - c*sin(e + f*x)
)), x)
```

3.18 $\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}(c-c\sin(e+fx))}\,dx$

3.18.1 Optimal result

Integrand size = 40, antiderivative size = 118

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}(c-c\sin(e+fx))}\,dx$$
$$= -\frac{\arctan\left(\frac{\sqrt{a}\sqrt{g}\cos(e+fx)}{\sqrt{2}\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}\right)}{\sqrt{2}\sqrt{a}cf\sqrt{g}} + \frac{\sec(e+fx)\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}{acfg}$$

output
```
-1/2*arctan(1/2*cos(f*x+e)*a^(1/2)*g^(1/2)*2^(1/2)/(g*sin(f*x+e))^(1/2)/(a
+a*sin(f*x+e))^(1/2))/c/f*2^(1/2)/a^(1/2)/g^(1/2)+sec(f*x+e)*(g*sin(f*x+e)
)^(1/2)*(a+a*sin(f*x+e))^(1/2)/a/c/f/g
```

3.18.2 Mathematica [A] (verified)

Time = 0.18 (sec) , antiderivative size = 132, normalized size of antiderivative = 1.12

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}(c-c\sin(e+fx))}\,dx$$
$$= \frac{\csc(2(e+fx))\sin^{\frac{3}{2}}(e+fx)\sqrt{a(1+\sin(e+fx))}\left(2\sqrt{c}\sqrt{\sin(e+fx)}+\sqrt{2}\arctan\left(\frac{\sqrt{2}\sqrt{c}\sqrt{\sin(e+fx)}}{\sqrt{c-c\sin(e+fx)}}\right)\sqrt{c-}\right.}{ac^{3/2}f\sqrt{g\sin(e+fx)}}$$

input
```
Integrate[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]]*(c - c*Sin[e +
f*x])),x]
```

output
```
(Csc[2*(e + f*x)]*Sin[e + f*x]^(3/2)*Sqrt[a*(1 + Sin[e + f*x])]*(2*Sqrt[c]
*Sqrt[Sin[e + f*x]] + Sqrt[2]*ArcTan[(Sqrt[2]*Sqrt[c]*Sqrt[Sin[e + f*x]])/
Sqrt[c - c*Sin[e + f*x]]]*Sqrt[c - c*Sin[e + f*x]]))/(a*c^(3/2)*f*Sqrt[g*S
in[e + f*x]])
```

3.18.3 Rubi [A] (verified)

Time = 0.87 (sec) , antiderivative size = 118, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}} = 0.175$, Rules used = {3042, 3417, 3042, 3261, 218, 3409, 15}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{1}{\sqrt{a\sin(e+fx)+a}(c-c\sin(e+fx))\sqrt{g\sin(e+fx)}}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{1}{\sqrt{a\sin(e+fx)+a}(c-c\sin(e+fx))\sqrt{g\sin(e+fx)}}dx$$

$$\Big\downarrow \text{3417}$$

$$\frac{\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{\sin(e+fx)a+a}}dx}{2c}+\frac{\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}(c-c\sin(e+fx))}dx}{2a}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{\sin(e+fx)a+a}}dx}{2c}+\frac{\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}(c-c\sin(e+fx))}dx}{2a}$$

$$\Big\downarrow \text{3261}$$

$$\frac{\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}(c-c\sin(e+fx))}dx}{2a}-\frac{a\int \frac{1}{\frac{\cos(e+fx)\cot(e+fx)a^3}{\sin(e+fx)a+a}+2a^2}d\frac{a\cos(e+fx)}{\sqrt{g\sin(e+fx)}\sqrt{\sin(e+fx)a+a}}}{cf}$$

$$\Big\downarrow \text{218}$$

$$\frac{\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}(c-c\sin(e+fx))}dx}{2a}-\frac{\arctan\left(\frac{\sqrt{a}\sqrt{g}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}\right)}{\sqrt{2}\sqrt{a}cf\sqrt{g}}$$

$$\Big\downarrow \text{3409}$$

$$-\frac{\int \frac{\sec(e+fx)(\sin(e+fx)a+a)\tan(e+fx)}{a^2c}d\frac{a\cos(e+fx)}{\sqrt{g\sin(e+fx)}\sqrt{\sin(e+fx)a+a}}}{f}-\frac{\arctan\left(\frac{\sqrt{a}\sqrt{g}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}\right)}{\sqrt{2}\sqrt{a}cf\sqrt{g}}$$

$$\Big\downarrow \text{ 15}$$

$$\frac{\sec(e+fx)\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}{acfg}-\frac{\arctan\left(\frac{\sqrt{a}\sqrt{g}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}\right)}{\sqrt{2}\sqrt{a}cf\sqrt{g}}$$

input
```
Int[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]]*(c - c*Sin[e + f*x]))
,x]
```

output
```
-(ArcTan[(Sqrt[a]*Sqrt[g]*Cos[e + f*x])/(Sqrt[2]*Sqrt[g*Sin[e + f*x]]*Sqrt
[a + a*Sin[e + f*x]])]/(Sqrt[2]*Sqrt[a]*c*f*Sqrt[g])) + (Sec[e + f*x]*Sqrt
[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]])/(a*c*f*g)
```

3.18.3.1 Defintions of rubi rules used

rule 15
```
Int[(a_.)*(x_)^(m_.), x_Symbol] :> Simp[a*(x^(m + 1)/(m + 1)), x] /; FreeQ[
{a, m}, x] && NeQ[m, -1]
```

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3261
```
Int[1/(Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(c_.) + (d_.)*sin[(e
_.) + (f_.)*(x_)]]), x_Symbol] :> Simp[-2*(a/f)   Subst[Int[1/(2*b^2 - (a*c
 - b*d)*x^2), x], x, b*(Cos[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*S
in[e + f*x]]))], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] &&
 EqQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]
```

rule 3409
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(g_.)*sin[(e_.) + (f_.
)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[-2*(b/f
)   Subst[Int[1/(b*c + a*d + c*g*x^2), x], x, b*(Cos[e + f*x]/(Sqrt[g*Sin[e
 + f*x]]*Sqrt[a + b*Sin[e + f*x]]))], x] /; FreeQ[{a, b, c, d, e, f, g}, x]
 && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0]
```

rule 3417
```
Int[1/(Sqrt[(g_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(a_) + (b_.)*sin[(e_.) + (f
_.)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[b/(b*
c - a*d)   Int[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]), x], x] -
Simp[d/(b*c - a*d)   Int[Sqrt[a + b*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c
+ d*Sin[e + f*x])), x], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c -
a*d, 0] && (EqQ[a^2 - b^2, 0] || EqQ[c^2 - d^2, 0])
```

3.18.4 Maple [A] (verified)

Time = 3.40 (sec) , antiderivative size = 130, normalized size of antiderivative = 1.10

method	result
default	$-\frac{\left(2\cos(fx+e)\sqrt{\csc(fx+e)-\cot(fx+e)}\,\arctan\left(\sqrt{\csc(fx+e)-\cot(fx+e)}\right)+1-\cos(fx+e)-\cos(fx+e)\cot(fx+e)+\csc(fx+e)\right)(1\cdots}{cf(-\cos(fx+e)+\sin(fx+e)-1)\sqrt{a(1+\sin(fx+e))}\,\sqrt{g\sin(fx+e)}}$

input
```
int(1/(c-c*sin(f*x+e))/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2),x,metho
d=_RETURNVERBOSE)
```

output
```
-1/c/f*(2*cos(f*x+e)*(csc(f*x+e)-cot(f*x+e))^(1/2)*arctan((csc(f*x+e)-cot(
f*x+e))^(1/2))+1-cos(f*x+e)-cos(f*x+e)*cot(f*x+e)+csc(f*x+e))*(1+cos(f*x+e
))/(-cos(f*x+e)+sin(f*x+e)-1)/(a*(1+sin(f*x+e)))^(1/2)/(g*sin(f*x+e))^(1/2
)
```

3.18.5 Fricas [A] (verification not implemented)

Time = 0.37 (sec) , antiderivative size = 391, normalized size of antiderivative = 3.31

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}(c-c\sin(e+fx))}\,dx$$

$$= \left[\frac{\sqrt{2}ag\sqrt{-\frac{1}{ag}}\cos(fx+e)\log\left(-\frac{4\sqrt{2}\left(3\cos(fx+e)^2+(3\cos(fx+e)+4)\sin(fx+e)-\cos(fx+e)-4\right)\sqrt{a\sin(fx+e)+a}\sqrt{g\sin(fx+e)\cdots}}{\cos(fx+e)^3+3\cos(fx+e)^2+\left(\cos(fx+e\cdots\right.}\right.}{8\cdots}\right.$$

input
```
integrate(1/(c-c*sin(f*x+e))/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2),x
, algorithm="fricas")
```

output
```
[1/8*(sqrt(2)*a*g*sqrt(-1/(a*g))*cos(f*x + e)*log(-(4*sqrt(2)*(3*cos(f*x +
 e)^2 + (3*cos(f*x + e) + 4)*sin(f*x + e) - cos(f*x + e) - 4)*sqrt(a*sin(f
*x + e) + a)*sqrt(g*sin(f*x + e))*sqrt(-1/(a*g)) - 17*cos(f*x + e)^3 - 3*c
os(f*x + e)^2 - (17*cos(f*x + e)^2 + 14*cos(f*x + e) - 4)*sin(f*x + e) + 1
8*cos(f*x + e) + 4)/(cos(f*x + e)^3 + 3*cos(f*x + e)^2 + (cos(f*x + e)^2 -
 2*cos(f*x + e) - 4)*sin(f*x + e) - 2*cos(f*x + e) - 4)) + 8*sqrt(a*sin(f*
x + e) + a)*sqrt(g*sin(f*x + e)))/(a*c*f*g*cos(f*x + e)), 1/4*(sqrt(2)*a*g
*sqrt(1/(a*g))*arctan(1/4*sqrt(2)*sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*x
+ e))*sqrt(1/(a*g))*(3*sin(f*x + e) - 1)/(cos(f*x + e)*sin(f*x + e)))*cos(
f*x + e) + 4*sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*x + e)))/(a*c*f*g*cos(f
*x + e))]
```

3.18.6 Sympy [F]

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}(c-c\sin(e+fx))}\,dx = -\frac{\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a\sin(e+fx)+a}\sin(e+fx)-\sqrt{g\sin(e+fx)}\sqrt{a\sin(e+fx)+a}}\,dx}{c}$$

input
```
integrate(1/(c-c*sin(f*x+e))/(g*sin(f*x+e))**(1/2)/(a+a*sin(f*x+e))**(1/2)
,x)
```

output
```
-Integral(1/(sqrt(g*sin(e + f*x))*sqrt(a*sin(e + f*x) + a)*sin(e + f*x) -
sqrt(g*sin(e + f*x))*sqrt(a*sin(e + f*x) + a)), x)/c
```

3.18.7 Maxima [F]

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}(c-c\sin(e+fx))}\,dx = \int -\frac{1}{\sqrt{a\sin(fx+e)+a}(c\sin(fx+e)-c)\sqrt{g\sin(fx+e)}}\,dx$$

input
```
integrate(1/(c-c*sin(f*x+e))/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2),x
, algorithm="maxima")
```

output
```
-integrate(1/(sqrt(a*sin(f*x + e) + a)*(c*sin(f*x + e) - c)*sqrt(g*sin(f*x
 + e))), x)
```

3.18.8 Giac [F(-1)]

Timed out.

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}(c-c\sin(e+fx))}\,dx = \text{Timed out}$$

input
```
integrate(1/(c-c*sin(f*x+e))/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2),x
, algorithm="giac")
```

output
```
Timed out
```

3.18.9 Mupad [F(-1)]

Timed out.

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}(c-c\sin(e+fx))}\,dx = \int \frac{1}{\sqrt{g\,\sin\left(e+f\,x\right)}\,\sqrt{a+a\,\sin\left(e+f\,x\right)}\,\left(c-c\,\sin\left(e+f\,x\right)\right)}\,dx$$

input
```
int(1/((g*sin(e + f*x))^(1/2)*(a + a*sin(e + f*x))^(1/2)*(c - c*sin(e + f*
x))),x)
```

output
```
int(1/((g*sin(e + f*x))^(1/2)*(a + a*sin(e + f*x))^(1/2)*(c - c*sin(e + f*
x))), x)
```

3.19 $\int \csc(e+fx)\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}\,dx$

3.19.1 Optimal result

Integrand size = 36, antiderivative size = 46

$$\int \csc(e+fx)\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}\,dx = \frac{\log(\sin(e+fx))\sec(e+fx)\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}}{f}$$

output
```
ln(sin(f*x+e))*sec(f*x+e)*(a+a*sin(f*x+e))^(1/2)*(c-c*sin(f*x+e))^(1/2)/f
```

3.19.2 Mathematica [A] (verified)

Time = 0.07 (sec) , antiderivative size = 62, normalized size of antiderivative = 1.35

$$\int \csc(e+fx)\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}\,dx = \frac{\left(\log\left(\cos\left(\frac{1}{2}(e+fx)\right)\right)+\log\left(\sin\left(\frac{1}{2}(e+fx)\right)\right)\right)\sec(e+fx)\sqrt{a(1+\sin(e+fx))}\sqrt{c-c\sin(e+fx)}}{f}$$

input
```
Integrate[Csc[e + f*x]*Sqrt[a + a*Sin[e + f*x]]*Sqrt[c - c*Sin[e + f*x]],x
]
```

output
```
((Log[Cos[(e + f*x)/2]] + Log[Sin[(e + f*x)/2]])*Sec[e + f*x]*Sqrt[a*(1 +
Sin[e + f*x])]*Sqrt[c - c*Sin[e + f*x]])/f
```

3.19.3 Rubi [A] (verified)

Time = 0.41 (sec) , antiderivative size = 48, normalized size of antiderivative = 1.04, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.139$, Rules used = {3042, 3427, 3042, 25, 3956}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \csc(e+fx)\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}}{\sin(e+fx)}dx$$

$$\Big\downarrow \text{3427}$$

$$\sec(e+fx)\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}\int \cot(e+fx)dx$$

$$\Big\downarrow \text{3042}$$

$$\sec(e+fx)\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}\int -\tan\left(e+fx+\frac{\pi}{2}\right)dx$$

$$\Big\downarrow \text{25}$$

$$-\sec(e+fx)\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}\int \tan\left(\frac{1}{2}(2e+\pi)+fx\right)dx$$

$$\Big\downarrow \text{3956}$$

$$\frac{\sec(e+fx)\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}\log(-\sin(e+fx))}{f}$$

input
```
Int[Csc[e + f*x]*Sqrt[a + a*Sin[e + f*x]]*Sqrt[c - c*Sin[e + f*x]],x]
```

output
```
(Log[-Sin[e + f*x]]*Sec[e + f*x]*Sqrt[a + a*Sin[e + f*x]]*Sqrt[c - c*Sin[e
 + f*x]])/f
```

3.19.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3427
```
Int[(Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(c_) + (d_.)*sin[(e_.)
 + (f_.)*(x_)]])/sin[(e_.) + (f_.)*(x_)], x_Symbol] :> Simp[Sqrt[a + b*Sin[
e + f*x]]*(Sqrt[c + d*Sin[e + f*x]]/Cos[e + f*x])   Int[Cot[e + f*x], x], x
] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0]
 && EqQ[c^2 - d^2, 0]
```

rule 3956
```
Int[tan[(c_.) + (d_.)*(x_)], x_Symbol] :> Simp[-Log[RemoveContent[Cos[c + d
*x], x]]/d, x] /; FreeQ[{c, d}, x]
```

3.19.4 Maple [A] (verified)

Time = 1.56 (sec) , antiderivative size = 68, normalized size of antiderivative = 1.48

method	result	size
default	$\frac{\sec(fx+e)\left(\ln(\csc(fx+e)-\cot(fx+e))-\ln\left(\frac{2}{1+\cos(fx+e)}\right)\right)\sqrt{a(1+\sin(fx+e))}\,\sqrt{-c(\sin(fx+e)-1)}}{f}$	68

input
```
int((a+a*sin(f*x+e))^(1/2)*(c-c*sin(f*x+e))^(1/2)/sin(f*x+e),x,method=_RET
URNVERBOSE)
```

output
```
1/f*sec(f*x+e)*(ln(csc(f*x+e)-cot(f*x+e))-ln(2/(1+cos(f*x+e))))*(a*(1+sin(
f*x+e)))^(1/2)*(-c*(sin(f*x+e)-1))^(1/2)
```

3.19.5 Fricas [A] (verification not implemented)

Time = 0.38 (sec) , antiderivative size = 202, normalized size of antiderivative = 4.39

$$\int \csc(e+fx)\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}\,dx$$
$$= \left[\frac{\sqrt{ac}\log\left(\frac{4\left(256\,ac\cos(fx+e)^5-512\,ac\cos(fx+e)^3+337\,ac\cos(fx+e)+\left(256\,\cos(fx+e)^4-512\,\cos(fx+e)^2+175\right)\sqrt{ac}\sqrt{a\sin(fx+e)+a}\sqrt{}\right.}{\cos(fx+e)^3-\cos(fx+e)}\right.}{2\,f}\right.$$
$$\left.-\frac{\sqrt{-ac}\arctan\left(\frac{\sqrt{-ac}\left(16\,\cos(fx+e)^2-7\right)\sqrt{a\sin(fx+e)+a}\sqrt{-c\sin(fx+e)+c}}{16\,ac\cos(fx+e)^3-25\,ac\cos(fx+e)}\right)}{f}\right]$$

input
```
integrate((a+a*sin(f*x+e))^(1/2)*(c-c*sin(f*x+e))^(1/2)/sin(f*x+e),x, algo
rithm="fricas")
```

output
```
[1/2*sqrt(a*c)*log(4*(256*a*c*cos(f*x + e)^5 - 512*a*c*cos(f*x + e)^3 + 33
7*a*c*cos(f*x + e) + (256*cos(f*x + e)^4 - 512*cos(f*x + e)^2 + 175)*sqrt(
a*c)*sqrt(a*sin(f*x + e) + a)*sqrt(-c*sin(f*x + e) + c))/(cos(f*x + e)^3 -
 cos(f*x + e)))/f, -sqrt(-a*c)*arctan(sqrt(-a*c)*(16*cos(f*x + e)^2 - 7)*s
qrt(a*sin(f*x + e) + a)*sqrt(-c*sin(f*x + e) + c)/(16*a*c*cos(f*x + e)^3 -
 25*a*c*cos(f*x + e)))/f]
```

3.19.6 Sympy [F]

$$\int \csc(e+fx)\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}\,dx$$
$$= \int \frac{\sqrt{a\left(\sin\left(e+fx\right)+1\right)}\sqrt{-c\left(\sin\left(e+fx\right)-1\right)}}{\sin\left(e+fx\right)}\,dx$$

input
```
integrate((a+a*sin(f*x+e))**(1/2)*(c-c*sin(f*x+e))**(1/2)/sin(f*x+e),x)
```

output
```
Integral(sqrt(a*(sin(e + f*x) + 1))*sqrt(-c*(sin(e + f*x) - 1))/sin(e + f*
x), x)
```

3.19.7 Maxima [F]

$$\int \csc(e+fx)\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}\,dx = \int \frac{\sqrt{a\sin(fx+e)+a}\sqrt{-c\sin(fx+e)+c}}{\sin(fx+e)}\,dx$$

input
```
integrate((a+a*sin(f*x+e))^(1/2)*(c-c*sin(f*x+e))^(1/2)/sin(f*x+e),x, algo
rithm="maxima")
```

output
```
integrate(sqrt(a*sin(f*x + e) + a)*sqrt(-c*sin(f*x + e) + c)/sin(f*x + e),
 x)
```

3.19.8 Giac [A] (verification not implemented)

Time = 0.34 (sec) , antiderivative size = 57, normalized size of antiderivative = 1.24

$$\int \csc(e+fx)\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}\,dx = -\frac{\sqrt{a}\sqrt{c}\log\left(\left|2\sin\left(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e\right)^2-1\right|\right)\operatorname{sgn}\left(\cos\left(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e\right)\right)\operatorname{sgn}\left(\sin\left(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e\right)\right)}{f}$$

input
```
integrate((a+a*sin(f*x+e))^(1/2)*(c-c*sin(f*x+e))^(1/2)/sin(f*x+e),x, algo
rithm="giac")
```

output
```
-sqrt(a)*sqrt(c)*log(abs(2*sin(-1/4*pi + 1/2*f*x + 1/2*e)^2 - 1))*sgn(cos(
-1/4*pi + 1/2*f*x + 1/2*e))*sgn(sin(-1/4*pi + 1/2*f*x + 1/2*e))/f
```

3.19.9 Mupad [F(-1)]

Timed out.

$$\int \csc(e+fx)\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}\,dx = \int \frac{\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}}{\sin(e+fx)}\,dx$$

input
```
int(((a + a*sin(e + f*x))^(1/2)*(c - c*sin(e + f*x))^(1/2))/sin(e + f*x),x
)
```

output
```
int(((a + a*sin(e + f*x))^(1/2)*(c - c*sin(e + f*x))^(1/2))/sin(e + f*x),
x)
```

3.19. $\int \csc(e+fx)\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}\,dx$

3.20 $\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c-c\sin(e+fx)}}\,dx$

3.20.1 Optimal result

Integrand size = 36, antiderivative size = 102

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c-c\sin(e+fx)}}\,dx = -\frac{a\cos(e+fx)\log(1-\sin(e+fx))}{f\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}} + \frac{\log(\sin(e+fx))\sec(e+fx)\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}}{cf}$$

output
```
-a*cos(f*x+e)*ln(1-sin(f*x+e))/f/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))^(
1/2)+ln(sin(f*x+e))*sec(f*x+e)*(a+a*sin(f*x+e))^(1/2)*(c-c*sin(f*x+e))^(1/
2)/c/f
```

3.20.2 Mathematica [A] (verified)

Time = 1.25 (sec) , antiderivative size = 64, normalized size of antiderivative = 0.63

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c-c\sin(e+fx)}}\,dx = -\frac{(\log(1-\sin(e+fx))-\log(\sin(e+fx)))\sec(e+fx)\sqrt{a(1+\sin(e+fx))}\sqrt{c-c\sin(e+fx)}}{cf}$$

input
```
Integrate[(Csc[e + f*x]*Sqrt[a + a*Sin[e + f*x]])/Sqrt[c - c*Sin[e + f*x]]
,x]
```

output
```
-(((Log[1 - Sin[e + f*x]] - Log[Sin[e + f*x]])*Sec[e + f*x]*Sqrt[a*(1 + Si
n[e + f*x])]*Sqrt[c - c*Sin[e + f*x]])/(c*f))
```

3.20.3 Rubi [A] (verified)

Time = 1.20 (sec) , antiderivative size = 105, normalized size of antiderivative = 1.03, number of steps used = 12, number of rules used = 11, $\frac{\text{number of rules}}{\text{integrand size}} = 0.306$, Rules used = {3042, 3421, 3042, 3216, 3042, 3146, 16, 3427, 3042, 25, 3956}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(e+fx)\sqrt{a\sin(e+fx)+a}}{\sqrt{c-c\sin(e+fx)}}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\sqrt{a\sin(e+fx)+a}}{\sin(e+fx)\sqrt{c-c\sin(e+fx)}}dx$$

$$\Big\downarrow \text{3421}$$

$$\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{c-c\sin(e+fx)}}dx+\frac{\int \csc(e+fx)\sqrt{\sin(e+fx)a+a}\sqrt{c-c\sin(e+fx)}dx}{c}$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{c-c\sin(e+fx)}}dx+\frac{\int \frac{\sqrt{\sin(e+fx)a+a}\sqrt{c-c\sin(e+fx)}}{\sin(e+fx)}dx}{c}$$

$$\Big\downarrow \text{3216}$$

$$\frac{\int \frac{\sqrt{\sin(e+fx)a+a}\sqrt{c-c\sin(e+fx)}}{\sin(e+fx)}dx}{c}+\frac{ac\cos(e+fx)\int \frac{\cos(e+fx)}{c-c\sin(e+fx)}dx}{\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{\sqrt{\sin(e+fx)a+a}\sqrt{c-c\sin(e+fx)}}{\sin(e+fx)}dx}{c}+\frac{ac\cos(e+fx)\int \frac{\cos(e+fx)}{c-c\sin(e+fx)}dx}{\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}}$$

$$\Big\downarrow \text{3146}$$

$$\frac{\int \frac{\sqrt{\sin(e+fx)a+a}\sqrt{c-c\sin(e+fx)}}{\sin(e+fx)}dx}{c}-\frac{a\cos(e+fx)\int \frac{1}{c-c\sin(e+fx)}d(-c\sin(e+fx))}{f\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}}$$

$$\Big\downarrow \text{16}$$

$$\frac{\int \frac{\sqrt{\sin(e+fx)a+a}\sqrt{c-c\sin(e+fx)}}{\sin(e+fx)}dx}{c}-\frac{a\cos(e+fx)\log(c-c\sin(e+fx))}{f\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}}$$

$$\Big\downarrow \text{3427}$$

$$\frac{\sec(e+fx)\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}\int \cot(e+fx)dx}{c}-\frac{a\cos(e+fx)\log(c-c\sin(e+fx))}{f\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\sec(e+fx)\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}\int -\tan\left(e+fx+\frac{\pi}{2}\right)dx}{c}-\frac{a\cos(e+fx)\log(c-c\sin(e+fx))}{f\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}}$$

$$\Big\downarrow \text{25}$$

$$-\frac{\sec(e+fx)\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}\int \tan\left(\frac{1}{2}(2e+\pi)+fx\right)dx}{c}-\frac{a\cos(e+fx)\log(c-c\sin(e+fx))}{f\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}}$$

$$\Big\downarrow \text{3956}$$

$$\frac{\sec(e+fx)\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}\log(-\sin(e+fx))}{cf}-\frac{a\cos(e+fx)\log(c-c\sin(e+fx))}{f\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}}$$

input
```
Int[(Csc[e + f*x]*Sqrt[a + a*Sin[e + f*x]])/Sqrt[c - c*Sin[e + f*x]],x]
```

output
```
-((a*Cos[e + f*x]*Log[c - c*Sin[e + f*x]])/(f*Sqrt[a + a*Sin[e + f*x]]*Sqr
t[c - c*Sin[e + f*x]])) + (Log[-Sin[e + f*x]]*Sec[e + f*x]*Sqrt[a + a*Sin[
e + f*x]]*Sqrt[c - c*Sin[e + f*x]])/(c*f)
```

3.20. $\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c-c\sin(e+fx)}}\,dx$

3.20.3.1 Defintions of rubi rules used

rule 16
```
Int[(c_.)/((a_.) + (b_.)*(x_)), x_Symbol] :> Simp[c*(Log[RemoveContent[a +
b*x, x]]/b), x] /; FreeQ[{a, b, c}, x]
```

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3146
```
Int[cos[(e_.) + (f_.)*(x_)]^(p_.)*((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m
_.), x_Symbol] :> Simp[1/(b^p*f)   Subst[Int[(a + x)^(m + (p - 1)/2)*(a - x
)^((p - 1)/2), x], x, b*Sin[e + f*x]], x] /; FreeQ[{a, b, e, f, m}, x] && I
ntegerQ[(p - 1)/2] && EqQ[a^2 - b^2, 0] && (GeQ[p, -1] ||  !IntegerQ[m + 1/
2])
```

rule 3216
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/Sqrt[(c_) + (d_.)*sin[(e_.)
+ (f_.)*(x_)]], x_Symbol] :> Simp[a*c*(Cos[e + f*x]/(Sqrt[a + b*Sin[e + f*x
]]*Sqrt[c + d*Sin[e + f*x]]))   Int[Cos[e + f*x]/(c + d*Sin[e + f*x]), x],
x] /; FreeQ[{a, b, c, d, e, f}, x] && EqQ[b*c + a*d, 0] && EqQ[a^2 - b^2, 0
]
```

rule 3421
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(sin[(e_.) + (f_.)*(x_)]*Sqr
t[(c_) + (d_.)*sin[(e_.) + (f_.)*(x_)]]), x_Symbol] :> Simp[-d/c   Int[Sqrt
[a + b*Sin[e + f*x]]/Sqrt[c + d*Sin[e + f*x]], x], x] + Simp[1/c   Int[Sqrt
[a + b*Sin[e + f*x]]*(Sqrt[c + d*Sin[e + f*x]]/Sin[e + f*x]), x], x] /; Fre
eQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0] && EqQ[
b*c + a*d, 0]
```

rule 3427
```
Int[(Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(c_) + (d_.)*sin[(e_.)
 + (f_.)*(x_)]])/sin[(e_.) + (f_.)*(x_)], x_Symbol] :> Simp[Sqrt[a + b*Sin[
e + f*x]]*(Sqrt[c + d*Sin[e + f*x]]/Cos[e + f*x])   Int[Cot[e + f*x], x], x
] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0]
 && EqQ[c^2 - d^2, 0]
```

3.20. $\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c-c\sin(e+fx)}}\,dx$

rule 3956
```
Int[tan[(c_.) + (d_.)*(x_)], x_Symbol] :> Simp[-Log[RemoveContent[Cos[c + d
*x], x]]/d, x] /; FreeQ[{c, d}, x]
```

3.20.4 Maple [A] (verified)

Time = 1.23 (sec) , antiderivative size = 100, normalized size of antiderivative = 0.98

method	result	size
default	$\frac{\left(2\ln\left(\csc\left(fx+e\right)-\cot\left(fx+e\right)-1\right)-\ln\left(\csc\left(fx+e\right)-\cot\left(fx+e\right)\right)\right)\sqrt{a\left(1+\sin\left(fx+e\right)\right)}\,\left(-\cos\left(fx+e\right)+\sin\left(fx+e\right)-1\right)}{f\left(\cos\left(fx+e\right)+\sin\left(fx+e\right)+1\right)\sqrt{-c\left(\sin\left(fx+e\right)-1\right)}}$	100

input
```
int((a+a*sin(f*x+e))^(1/2)/sin(f*x+e)/(c-c*sin(f*x+e))^(1/2),x,method=_RET
URNVERBOSE)
```

output
```
1/f*(2*ln(csc(f*x+e)-cot(f*x+e)-1)-ln(csc(f*x+e)-cot(f*x+e)))*(a*(1+sin(f*
x+e)))^(1/2)*(-cos(f*x+e)+sin(f*x+e)-1)/(cos(f*x+e)+sin(f*x+e)+1)/(-c*(sin
(f*x+e)-1))^(1/2)
```

3.20.5 Fricas [F]

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c-c\sin(e+fx)}}\,dx=\int \frac{\sqrt{a\sin\left(fx+e\right)+a}}{\sqrt{-c\sin\left(fx+e\right)+c}\sin\left(fx+e\right)}\,dx$$

input
```
integrate((a+a*sin(f*x+e))^(1/2)/sin(f*x+e)/(c-c*sin(f*x+e))^(1/2),x, algo
rithm="fricas")
```

output
```
integral(sqrt(a*sin(f*x + e) + a)*sqrt(-c*sin(f*x + e) + c)/(c*cos(f*x + e
)^2 + c*sin(f*x + e) - c), x)
```

3.20.6 Sympy [F]

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c-c\sin(e+fx)}}\,dx=\int \frac{\sqrt{a\left(\sin\left(e+fx\right)+1\right)}}{\sqrt{-c\left(\sin\left(e+fx\right)-1\right)}\sin\left(e+fx\right)}\,dx$$

input
```
integrate((a+a*sin(f*x+e))**(1/2)/sin(f*x+e)/(c-c*sin(f*x+e))**(1/2),x)
```

output
```
Integral(sqrt(a*(sin(e + f*x) + 1))/(sqrt(-c*(sin(e + f*x) - 1))*sin(e + f
*x)), x)
```

3.20.7 Maxima [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 59, normalized size of antiderivative = 0.58

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c-c\sin(e+fx)}}\,dx=\frac{\frac{2\sqrt{a}\log\left(\frac{\sin(fx+e)}{\cos(fx+e)+1}-1\right)}{\sqrt{c}}-\frac{\sqrt{a}\log\left(\frac{\sin(fx+e)}{\cos(fx+e)+1}\right)}{\sqrt{c}}}{f}$$

input
```
integrate((a+a*sin(f*x+e))^(1/2)/sin(f*x+e)/(c-c*sin(f*x+e))^(1/2),x, algo
rithm="maxima")
```

output
```
(2*sqrt(a)*log(sin(f*x + e)/(cos(f*x + e) + 1) - 1)/sqrt(c) - sqrt(a)*log(
sin(f*x + e)/(cos(f*x + e) + 1))/sqrt(c))/f
```

3.20.8 Giac [A] (verification not implemented)

Time = 0.34 (sec) , antiderivative size = 1, normalized size of antiderivative = 0.01

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c-c\sin(e+fx)}}\,dx=0$$

input
```
integrate((a+a*sin(f*x+e))^(1/2)/sin(f*x+e)/(c-c*sin(f*x+e))^(1/2),x, algo
rithm="giac")
```

output
```
0
```

3.20.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c-c\sin(e+fx)}}\,dx = \int \frac{\sqrt{a+a\,\sin\left(e+f\,x\right)}}{\sin\left(e+f\,x\right)\,\sqrt{c-c\,\sin\left(e+f\,x\right)}}\,dx$$

input
```
int((a + a*sin(e + f*x))^(1/2)/(sin(e + f*x)*(c - c*sin(e + f*x))^(1/2)),x
)
```

output
```
int((a + a*sin(e + f*x))^(1/2)/(sin(e + f*x)*(c - c*sin(e + f*x))^(1/2)),
x)
```

3.21 $\int \frac{\csc(e+fx)\sqrt{c-c\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}}\,dx$

3.21.1 Optimal result

Integrand size = 36, antiderivative size = 100

$$\int \frac{\csc(e+fx)\sqrt{c-c\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}}\,dx = -\frac{c\cos(e+fx)\log(1+\sin(e+fx))}{f\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}} + \frac{\log(\sin(e+fx))\sec(e+fx)\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}}{af}$$

output
```
-c*cos(f*x+e)*ln(1+sin(f*x+e))/f/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))^(
1/2)+ln(sin(f*x+e))*sec(f*x+e)*(a+a*sin(f*x+e))^(1/2)*(c-c*sin(f*x+e))^(1/
2)/a/f
```

3.21.2 Mathematica [A] (verified)

Time = 0.77 (sec) , antiderivative size = 61, normalized size of antiderivative = 0.61

$$\int \frac{\csc(e+fx)\sqrt{c-c\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}}\,dx = \frac{(\log(\sin(e+fx))-\log(1+\sin(e+fx)))\sec(e+fx)\sqrt{a(1+\sin(e+fx))}\sqrt{c-c\sin(e+fx)}}{af}$$

input
```
Integrate[(Csc[e + f*x]*Sqrt[c - c*Sin[e + f*x]])/Sqrt[a + a*Sin[e + f*x]]
,x]
```

output
```
((Log[Sin[e + f*x]] - Log[1 + Sin[e + f*x]])*Sec[e + f*x]*Sqrt[a*(1 + Sin[
e + f*x])]*Sqrt[c - c*Sin[e + f*x]])/(a*f)
```

3.21.3 Rubi [A] (verified)

Time = 1.18 (sec) , antiderivative size = 104, normalized size of antiderivative = 1.04, number of steps used = 12, number of rules used = 11, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.306, Rules used = {3042, 3421, 3042, 3216, 3042, 3146, 16, 3427, 3042, 25, 3956}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(e+fx)\sqrt{c-c\sin(e+fx)}}{\sqrt{a\sin(e+fx)+a}}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\sqrt{c-c\sin(e+fx)}}{\sin(e+fx)\sqrt{a\sin(e+fx)+a}}dx$$

$$\Big\downarrow \text{3421}$$

$$\frac{\int \csc(e+fx)\sqrt{\sin(e+fx)a+a}\sqrt{c-c\sin(e+fx)}dx}{a}-\int \frac{\sqrt{c-c\sin(e+fx)}}{\sqrt{\sin(e+fx)a+a}}dx$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{\sqrt{\sin(e+fx)a+a}\sqrt{c-c\sin(e+fx)}}{\sin(e+fx)}dx}{a}-\int \frac{\sqrt{c-c\sin(e+fx)}}{\sqrt{\sin(e+fx)a+a}}dx$$

$$\Big\downarrow \text{3216}$$

$$\frac{\int \frac{\sqrt{\sin(e+fx)a+a}\sqrt{c-c\sin(e+fx)}}{\sin(e+fx)}dx}{a}-\frac{ac\cos(e+fx)\int \frac{\cos(e+fx)}{\sin(e+fx)a+a}dx}{\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{\sqrt{\sin(e+fx)a+a}\sqrt{c-c\sin(e+fx)}}{\sin(e+fx)}dx}{a}-\frac{ac\cos(e+fx)\int \frac{\cos(e+fx)}{\sin(e+fx)a+a}dx}{\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}}$$

$$\Big\downarrow \text{3146}$$

$$\frac{\int \frac{\sqrt{\sin(e+fx)a+a}\sqrt{c-c\sin(e+fx)}}{\sin(e+fx)}dx}{a}-\frac{c\cos(e+fx)\int \frac{1}{\sin(e+fx)a+a}d(a\sin(e+fx))}{f\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}}$$

$$\Big\downarrow \text{16}$$

$$\frac{\int \frac{\sqrt{\sin(e+fx)a+a}\sqrt{c-c\sin(e+fx)}}{\sin(e+fx)}dx}{a}-\frac{c\cos(e+fx)\log(a\sin(e+fx)+a)}{f\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}}$$

$$\Big\downarrow \text{3427}$$

$$\frac{\sec(e+fx)\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}\int \cot(e+fx)dx}{a}-\frac{c\cos(e+fx)\log(a\sin(e+fx)+a)}{f\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\sec(e+fx)\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}\int -\tan\left(e+fx+\frac{\pi}{2}\right)dx}{a}-\frac{c\cos(e+fx)\log(a\sin(e+fx)+a)}{f\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}}$$

$$\Big\downarrow \text{25}$$

$$-\frac{\sec(e+fx)\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}\int \tan\left(\frac{1}{2}(2e+\pi)+fx\right)dx}{a}-\frac{c\cos(e+fx)\log(a\sin(e+fx)+a)}{f\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}}$$

$$\Big\downarrow \text{3956}$$

$$\frac{\sec(e+fx)\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}\log(-\sin(e+fx))}{af}-\frac{c\cos(e+fx)\log(a\sin(e+fx)+a)}{f\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}}$$

input
```
Int[(Csc[e + f*x]*Sqrt[c - c*Sin[e + f*x]])/Sqrt[a + a*Sin[e + f*x]],x]
```

output
```
-((c*Cos[e + f*x]*Log[a + a*Sin[e + f*x]])/(f*Sqrt[a + a*Sin[e + f*x]]*Sqr
t[c - c*Sin[e + f*x]])) + (Log[-Sin[e + f*x]]*Sec[e + f*x]*Sqrt[a + a*Sin[
e + f*x]]*Sqrt[c - c*Sin[e + f*x]])/(a*f)
```

3.21. $\int \frac{\csc(e+fx)\sqrt{c-c\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}}dx$

3.21.3.1 Defintions of rubi rules used

rule 16
```
Int[(c_.)/((a_.) + (b_.)*(x_)), x_Symbol] :> Simp[c*(Log[RemoveContent[a +
b*x, x]]/b), x] /; FreeQ[{a, b, c}, x]
```

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3146
```
Int[cos[(e_.) + (f_.)*(x_)]^(p_.)*((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m
_.), x_Symbol] :> Simp[1/(b^p*f)   Subst[Int[(a + x)^(m + (p - 1)/2)*(a - x
)^((p - 1)/2), x], x, b*Sin[e + f*x]], x] /; FreeQ[{a, b, e, f, m}, x] && I
ntegerQ[(p - 1)/2] && EqQ[a^2 - b^2, 0] && (GeQ[p, -1] ||  !IntegerQ[m + 1/
2])
```

rule 3216
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/Sqrt[(c_) + (d_.)*sin[(e_.)
+ (f_.)*(x_)]], x_Symbol] :> Simp[a*c*(Cos[e + f*x]/(Sqrt[a + b*Sin[e + f*x
]]*Sqrt[c + d*Sin[e + f*x]]))   Int[Cos[e + f*x]/(c + d*Sin[e + f*x]), x],
x] /; FreeQ[{a, b, c, d, e, f}, x] && EqQ[b*c + a*d, 0] && EqQ[a^2 - b^2, 0
]
```

rule 3421
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(sin[(e_.) + (f_.)*(x_)]*Sqr
t[(c_) + (d_.)*sin[(e_.) + (f_.)*(x_)]]), x_Symbol] :> Simp[-d/c   Int[Sqrt
[a + b*Sin[e + f*x]]/Sqrt[c + d*Sin[e + f*x]], x], x] + Simp[1/c   Int[Sqrt
[a + b*Sin[e + f*x]]*(Sqrt[c + d*Sin[e + f*x]]/Sin[e + f*x]), x], x] /; Fre
eQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0] && EqQ[
b*c + a*d, 0]
```

rule 3427
```
Int[(Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(c_) + (d_.)*sin[(e_.)
 + (f_.)*(x_)]])/sin[(e_.) + (f_.)*(x_)], x_Symbol] :> Simp[Sqrt[a + b*Sin[
e + f*x]]*(Sqrt[c + d*Sin[e + f*x]]/Cos[e + f*x])   Int[Cot[e + f*x], x], x
] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0]
 && EqQ[c^2 - d^2, 0]
```

3.21. $\int \frac{\csc(e+fx)\sqrt{c-c\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}}\,dx$

rule 3956
```
Int[tan[(c_.) + (d_.)*(x_)], x_Symbol] :> Simp[-Log[RemoveContent[Cos[c + d
*x], x]]/d, x] /; FreeQ[{c, d}, x]
```

3.21.4 Maple [A] (verified)

Time = 1.24 (sec) , antiderivative size = 100, normalized size of antiderivative = 1.00

method	result	size
default	$\frac{(2\ln(-\cot(fx+e)+\csc(fx+e)+1)-\ln(\csc(fx+e)-\cot(fx+e)))\sqrt{-c(\sin(fx+e)-1)}\,(\cos(fx+e)+\sin(fx+e)+1)}{f(-\cos(fx+e)+\sin(fx+e)-1)\sqrt{a(1+\sin(fx+e))}}$	100

input
```
int((c-c*sin(f*x+e))^(1/2)/sin(f*x+e)/(a+a*sin(f*x+e))^(1/2),x,method=_RET
URNVERBOSE)
```

output
```
1/f*(2*ln(-cot(f*x+e)+csc(f*x+e)+1)-ln(csc(f*x+e)-cot(f*x+e)))*(-c*(sin(f*
x+e)-1))^(1/2)*(cos(f*x+e)+sin(f*x+e)+1)/(-cos(f*x+e)+sin(f*x+e)-1)/(a*(1+
sin(f*x+e)))^(1/2)
```

3.21.5 Fricas [F]

$$\int \frac{\csc(e+fx)\sqrt{c-c\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}}\,dx = \int \frac{\sqrt{-c\sin(fx+e)+c}}{\sqrt{a\sin(fx+e)+a}\sin(fx+e)}\,dx$$

input
```
integrate((c-c*sin(f*x+e))^(1/2)/sin(f*x+e)/(a+a*sin(f*x+e))^(1/2),x, algo
rithm="fricas")
```

output
```
integral(-sqrt(a*sin(f*x + e) + a)*sqrt(-c*sin(f*x + e) + c)/(a*cos(f*x +
e)^2 - a*sin(f*x + e) - a), x)
```

3.21.6 Sympy [F]

$$\int \frac{\csc(e+fx)\sqrt{c-c\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}}\,dx = \int \frac{\sqrt{-c\left(\sin\left(e+fx\right)-1\right)}}{\sqrt{a\left(\sin\left(e+fx\right)+1\right)}\sin\left(e+fx\right)}\,dx$$

input
```
integrate((c-c*sin(f*x+e))**(1/2)/sin(f*x+e)/(a+a*sin(f*x+e))**(1/2),x)
```

output
```
Integral(sqrt(-c*(sin(e + f*x) - 1))/(sqrt(a*(sin(e + f*x) + 1))*sin(e + f
*x)), x)
```

3.21.7 Maxima [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 59, normalized size of antiderivative = 0.59

$$\int \frac{\csc(e+fx)\sqrt{c-c\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}}\,dx = \frac{\frac{2\sqrt{c}\log\left(\frac{\sin(fx+e)}{\cos(fx+e)+1}+1\right)}{\sqrt{a}} - \frac{\sqrt{c}\log\left(\frac{\sin(fx+e)}{\cos(fx+e)+1}\right)}{\sqrt{a}}}{f}$$

input
```
integrate((c-c*sin(f*x+e))^(1/2)/sin(f*x+e)/(a+a*sin(f*x+e))^(1/2),x, algo
rithm="maxima")
```

output
```
(2*sqrt(c)*log(sin(f*x + e)/(cos(f*x + e) + 1) + 1)/sqrt(a) - sqrt(c)*log(
sin(f*x + e)/(cos(f*x + e) + 1))/sqrt(a))/f
```

3.21.8 Giac [A] (verification not implemented)

Time = 0.34 (sec) , antiderivative size = 112, normalized size of antiderivative = 1.12

$$\int \frac{\csc(e+fx)\sqrt{c-c\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}}\,dx = \frac{\sqrt{2}\sqrt{a}\sqrt{c}\left(\frac{\sqrt{2}\log\left(-\sin\left(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e\right)^2+1\right)}{a\mathrm{sgn}\left(\cos\left(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e\right)\right)} - \frac{\sqrt{2}\log\left(\left|2\sin\left(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e\right)^2-1\right|\right)}{a\mathrm{sgn}\left(\cos\left(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e\right)\right)}\right)\mathrm{sgn}\left(\sin\left(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e\right)\right)}{2f}$$

input
```
integrate((c-c*sin(f*x+e))^(1/2)/sin(f*x+e)/(a+a*sin(f*x+e))^(1/2),x, algo
rithm="giac")
```

output
```
1/2*sqrt(2)*sqrt(a)*sqrt(c)*(sqrt(2)*log(-sin(-1/4*pi + 1/2*f*x + 1/2*e)^2
 + 1)/(a*sgn(cos(-1/4*pi + 1/2*f*x + 1/2*e))) - sqrt(2)*log(abs(2*sin(-1/4
*pi + 1/2*f*x + 1/2*e)^2 - 1))/(a*sgn(cos(-1/4*pi + 1/2*f*x + 1/2*e))))*sg
n(sin(-1/4*pi + 1/2*f*x + 1/2*e))/f
```

3.21.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)\sqrt{c-c\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}}\,dx=\int \frac{\sqrt{c-c\,\sin\left(e+f\,x\right)}}{\sin\left(e+f\,x\right)\,\sqrt{a+a\,\sin\left(e+f\,x\right)}}\,dx$$

input
```
int((c - c*sin(e + f*x))^(1/2)/(sin(e + f*x)*(a + a*sin(e + f*x))^(1/2)),x
)
```

output
```
int((c - c*sin(e + f*x))^(1/2)/(sin(e + f*x)*(a + a*sin(e + f*x))^(1/2)),
x)
```

3.22 $\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}}\,dx$

3.22.1 Optimal result

Integrand size = 36, antiderivative size = 46

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}}\,dx = \frac{\cos(e+fx)\log(\tan(e+fx))}{f\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}}$$

output
```
cos(f*x+e)*ln(tan(f*x+e))/f/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))^(1/2)
```

3.22.2 Mathematica [A] (verified)

Time = 0.11 (sec) , antiderivative size = 63, normalized size of antiderivative = 1.37

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}}\,dx = -\frac{(\log(\cos(e+fx))-\log(\sin(e+fx)))\sec(e+fx)\sqrt{a(1+\sin(e+fx))}\sqrt{c-c\sin(e+fx)}}{acf}$$

input
```
Integrate[Csc[e + f*x]/(Sqrt[a + a*Sin[e + f*x]]*Sqrt[c - c*Sin[e + f*x]])
,x]
```

output
```
-(((Log[Cos[e + f*x]] - Log[Sin[e + f*x]])*Sec[e + f*x]*Sqrt[a*(1 + Sin[e
+ f*x])]*Sqrt[c - c*Sin[e + f*x]])/(a*c*f))
```

3.22.3 Rubi [A] (verified)

Time = 0.41 (sec) , antiderivative size = 46, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.139$, Rules used = {3042, 3425, 3042, 3100, 14}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(e+fx)}{\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{1}{\sin(e+fx)\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}}dx$$

$$\Big\downarrow \text{3425}$$

$$\frac{\cos(e+fx)\int \csc(e+fx)\sec(e+fx)dx}{\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\cos(e+fx)\int \csc(e+fx)\sec(e+fx)dx}{\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}}$$

$$\Big\downarrow \text{3100}$$

$$\frac{\cos(e+fx)\int \cot(e+fx)d\tan(e+fx)}{f\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}}$$

$$\Big\downarrow \text{14}$$

$$\frac{\cos(e+fx)\log(\tan(e+fx))}{f\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}}$$

input
```
Int[Csc[e + f*x]/(Sqrt[a + a*Sin[e + f*x]]*Sqrt[c - c*Sin[e + f*x]]),x]
```

output
```
(Cos[e + f*x]*Log[Tan[e + f*x]])/(f*Sqrt[a + a*Sin[e + f*x]]*Sqrt[c - c*Si
n[e + f*x]])
```

3.22.3.1 Defintions of rubi rules used

rule 14
```
Int[(a_.)/(x_), x_Symbol] :> Simp[a*Log[x], x] /; FreeQ[a, x]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3100
```
Int[csc[(e_.) + (f_.)*(x_)]^(m_.)*sec[(e_.) + (f_.)*(x_)]^(n_.), x_Symbol]
:> Simp[1/f   Subst[Int[(1 + x^2)^((m + n)/2 - 1)/x^m, x], x, Tan[e + f*x]]
, x] /; FreeQ[{e, f}, x] && IntegersQ[m, n, (m + n)/2]
```

rule 3425
```
Int[1/(sin[(e_.) + (f_.)*(x_)]*Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]*S
qrt[(c_) + (d_.)*sin[(e_.) + (f_.)*(x_)]]), x_Symbol] :> Simp[Cos[e + f*x]/
(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]])   Int[1/(Cos[e + f*x]*S
in[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] &
& EqQ[a^2 - b^2, 0] && EqQ[c^2 - d^2, 0]
```

3.22.4 Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 90 vs. $2(42) = 84$.

Time = 1.79 (sec) , antiderivative size = 91, normalized size of antiderivative = 1.98

method	result	size
default	$\frac{\cos(fx+e)(\ln(\csc(fx+e)-\cot(fx+e))-\ln(\csc(fx+e)-\cot(fx+e)-1)-\ln(-\cot(fx+e)+\csc(fx+e)+1))}{f\sqrt{a(1+\sin(fx+e))}\,\sqrt{-c(\sin(fx+e)-1)}}$	91

input
```
int(1/sin(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))^(1/2),x,method=_R
ETURNVERBOSE)
```

output
```
1/f*cos(f*x+e)*(ln(csc(f*x+e)-cot(f*x+e))-ln(csc(f*x+e)-cot(f*x+e)-1)-ln(-
cot(f*x+e)+csc(f*x+e)+1))/(a*(1+sin(f*x+e)))^(1/2)/(-c*(sin(f*x+e)-1))^(1/
2)
```

3.22.5 Fricas [A] (verification not implemented)

Time = 0.33 (sec) , antiderivative size = 193, normalized size of antiderivative = 4.20

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}}\,dx = \left[\frac{\sqrt{ac}\log\left(-\frac{4\left(2\,ac\cos(fx+e)^5-2\,ac\cos(fx+e)^3+ac\cos(fx+e)-\sqrt{ac}\left(2\,\cos(fx+e)^2-1\right)\sqrt{a\sin(fx+e)+a}\sqrt{-c\sin(fx+e)+c}\right)}{\cos(fx+e)^5-\cos(fx+e)^3}\right)}{2\,acf}, \sqrt{-c\right.$$

input
```
integrate(1/sin(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))^(1/2),x, al
gorithm="fricas")
```

output
```
[1/2*sqrt(a*c)*log(-4*(2*a*c*cos(f*x + e)^5 - 2*a*c*cos(f*x + e)^3 + a*c*c
os(f*x + e) - sqrt(a*c)*(2*cos(f*x + e)^2 - 1)*sqrt(a*sin(f*x + e) + a)*sq
rt(-c*sin(f*x + e) + c))/(cos(f*x + e)^5 - cos(f*x + e)^3))/(a*c*f), sqrt(
-a*c)*arctan(sqrt(-a*c)*sqrt(a*sin(f*x + e) + a)*sqrt(-c*sin(f*x + e) + c)
/(2*a*c*cos(f*x + e)^3 - a*c*cos(f*x + e)))/(a*c*f)]
```

3.22.6 Sympy [F]

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}}\,dx = \int \frac{1}{\sqrt{a\left(\sin\left(e+fx\right)+1\right)}\sqrt{-c\left(\sin\left(e+fx\right)-1\right)}\sin\left(e+fx\right)}\,dx$$

input
```
integrate(1/sin(f*x+e)/(a+a*sin(f*x+e))**(1/2)/(c-c*sin(f*x+e))**(1/2),x)
```

output
```
Integral(1/(sqrt(a*(sin(e + f*x) + 1))*sqrt(-c*(sin(e + f*x) - 1))*sin(e +
 f*x)), x)
```

3.22.7 Maxima [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 0.40 (sec) , antiderivative size = 190, normalized size of antiderivative = 4.13

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}}\,dx = -\frac{(-1)^{4\cos(2fx+2e)}\cosh(4\pi\sin(2fx+2e))\log\left(\frac{16\left(\cos(2fx+2e)^2+\sin(2fx+2e)^2+2\cos(2fx+2e)+1\right)}{ac\left|e^{(2ifx+2ie)}-1\right|^2}\right)-2i(-1)^4}{2\sqrt{a}\sqrt{c}f}$$

input
```
integrate(1/sin(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))^(1/2),x, al
gorithm="maxima")
```

output
```
-1/2*((-1)^(4*cos(2*f*x + 2*e))*cosh(4*pi*sin(2*f*x + 2*e))*log(16*(cos(2*
f*x + 2*e)^2 + sin(2*f*x + 2*e)^2 + 2*cos(2*f*x + 2*e) + 1)/(a*c*abs(e^(2*
I*f*x + 2*I*e) - 1)^2)) - 2*I*(-1)^(4*cos(2*f*x + 2*e))*arctan2(4*sin(2*f*
x + 2*e)/(sqrt(a)*sqrt(c)*abs(e^(2*I*f*x + 2*I*e) - 1)), 4*(cos(2*f*x + 2*
e) + 1)/(sqrt(a)*sqrt(c)*abs(e^(2*I*f*x + 2*I*e) - 1)))*sinh(4*pi*sin(2*f*
x + 2*e)))/(sqrt(a)*sqrt(c)*f)
```

3.22.8 Giac [F(-2)]

Exception generated.

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}}\,dx = \text{Exception raised: TypeError}$$

input
```
integrate(1/sin(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))^(1/2),x, al
gorithm="giac")
```

output
```
Exception raised: TypeError >> an error occurred running a Giac command:IN
PUT:sage2:=int(sage0,sageVARx):;OUTPUT:index.cc index_m i_lex_is_greater E
rror: Bad Argument Value
```

3.22.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}}\,dx$$
$$=\int \frac{1}{\sin\left(e+f\,x\right)\,\sqrt{a+a\,\sin\left(e+f\,x\right)}\,\sqrt{c-c\,\sin\left(e+f\,x\right)}}\,dx$$

input
```
int(1/(sin(e + f*x)*(a + a*sin(e + f*x))^(1/2)*(c - c*sin(e + f*x))^(1/2))
,x)
```

output
```
int(1/(sin(e + f*x)*(a + a*sin(e + f*x))^(1/2)*(c - c*sin(e + f*x))^(1/2))
, x)
```

3.23 $\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{c+d\sin(e+fx)}\,dx$

3.23.1 Optimal result

Integrand size = 33, antiderivative size = 105

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{c+d\sin(e+fx)}\,dx = -\frac{2\sqrt{a}\,\text{arctanh}\left(\frac{\sqrt{a}\cos(e+fx)}{\sqrt{a+a\sin(e+fx)}}\right)}{cf} + \frac{2\sqrt{a}\sqrt{d}\,\text{arctanh}\left(\frac{\sqrt{a}\sqrt{d}\cos(e+fx)}{\sqrt{c+d}\sqrt{a+a\sin(e+fx)}}\right)}{c\sqrt{c+d}\,f}$$

output
```
-2*arctanh(cos(f*x+e)*a^(1/2)/(a+a*sin(f*x+e))^(1/2))*a^(1/2)/c/f+2*arctan
h(cos(f*x+e)*a^(1/2)*d^(1/2)/(c+d)^(1/2)/(a+a*sin(f*x+e))^(1/2))*a^(1/2)*d
^(1/2)/c/f/(c+d)^(1/2)
```

3.23.2 Mathematica [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.

Time = 6.01 (sec) , antiderivative size = 746, normalized size of antiderivative = 7.10

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{c+d\sin(e+fx)}\,dx = -\frac{\left(\frac{1}{8}-\frac{i}{8}\right)\left((4+4i)\sqrt{c+d}\left(\log\left(1+\cos\left(\frac{1}{2}(e+fx)\right)-\sin\left(\frac{1}{2}(e+fx)\right)\right)-\log\left(1-\cos\left(\frac{1}{2}(e+fx)\right)+\sin\right.\right.\right.}{}$$

input
```
Integrate[(Csc[e + f*x]*Sqrt[a + a*Sin[e + f*x]])/(c + d*Sin[e + f*x]),x]
```

output
```
((-1/8 + I/8)*((4 + 4*I)*Sqrt[c + d]*(Log[1 + Cos[(e + f*x)/2] - Sin[(e +
f*x)/2]] - Log[1 - Cos[(e + f*x)/2] + Sin[(e + f*x)/2]]) + Sqrt[d]*RootSum
[-d + (2*I)*c*E^(I*e)*#1^2 + d*E^((2*I)*e)*#1^4 & , ((1 + I)*d*Sqrt[E^((-I
)*e)]*f*x - (2 - 2*I)*d*Sqrt[E^((-I)*e)]*Log[E^((I/2)*f*x) - #1] - I*Sqrt[
d]*Sqrt[c + d]*f*x*#1 + 2*Sqrt[d]*Sqrt[c + d]*Log[E^((I/2)*f*x) - #1]*#1 +
 ((1 - I)*c*f*x*#1^2)/Sqrt[E^((-I)*e)] + ((2 + 2*I)*c*Log[E^((I/2)*f*x) -
#1]*#1^2)/Sqrt[E^((-I)*e)] - Sqrt[d]*Sqrt[c + d]*E^(I*e)*f*x*#1^3 - (2*I)*
Sqrt[d]*Sqrt[c + d]*E^(I*e)*Log[E^((I/2)*f*x) - #1]*#1^3)/((-I)*d - c*E^(I
*e)*#1^2) & ]*(Cos[e/2] + I*Sin[e/2]) + Sqrt[d]*RootSum[-d + (2*I)*c*E^(I*
e)*#1^2 + d*E^((2*I)*e)*#1^4 & , ((1 - I)*d*Sqrt[E^((-I)*e)]*f*x + (2 + 2*
I)*d*Sqrt[E^((-I)*e)]*Log[E^((I/2)*f*x) - #1] + Sqrt[d]*Sqrt[c + d]*f*x*#1
 + (2*I)*Sqrt[d]*Sqrt[c + d]*Log[E^((I/2)*f*x) - #1]*#1 - ((1 + I)*c*f*x*#
1^2)/Sqrt[E^((-I)*e)] + ((2 - 2*I)*c*Log[E^((I/2)*f*x) - #1]*#1^2)/Sqrt[E^
((-I)*e)] - I*Sqrt[d]*Sqrt[c + d]*E^(I*e)*f*x*#1^3 + 2*Sqrt[d]*Sqrt[c + d]
*E^(I*e)*Log[E^((I/2)*f*x) - #1]*#1^3)/(d - I*c*E^(I*e)*#1^2) & ]*(Cos[e/2
] + I*Sin[e/2]))*Sqrt[a*(1 + Sin[e + f*x])])/(c*Sqrt[c + d]*f*(Cos[(e + f*
x)/2] + Sin[(e + f*x)/2]))
```

3.23.3 Rubi [A] (verified)

Time = 0.54 (sec) , antiderivative size = 105, normalized size of antiderivative = 1.00, number of steps used = 7, number of rules used = 6, $\frac{\text{number of rules}}{\text{integrand size}} = 0.182$, Rules used = {3042, 3413, 3042, 3252, 219, 221}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(e+fx)\sqrt{a\sin(e+fx)+a}}{c+d\sin(e+fx)}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\sqrt{a\sin(e+fx)+a}}{\sin(e+fx)(c+d\sin(e+fx))}dx$$

$$\Big\downarrow \text{3413}$$

$$\frac{\int \csc(e+fx)\sqrt{\sin(e+fx)a+a}dx}{c}-\frac{d\int \frac{\sqrt{\sin(e+fx)a+a}}{c+d\sin(e+fx)}dx}{c}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)}dx}{c} - \frac{d\int \frac{\sqrt{\sin(e+fx)a+a}}{c+d\sin(e+fx)}dx}{c}$$

$$\Big\downarrow \text{3252}$$

$$\frac{2ad\int \frac{1}{a(c+d)-\frac{a^2d\cos^2(e+fx)}{\sin(e+fx)a+a}}d\frac{a\cos(e+fx)}{\sqrt{\sin(e+fx)a+a}}}{cf} - \frac{2a\int \frac{1}{a-\frac{a^2\cos^2(e+fx)}{\sin(e+fx)a+a}}d\frac{a\cos(e+fx)}{\sqrt{\sin(e+fx)a+a}}}{cf}$$

$$\Big\downarrow \text{219}$$

$$\frac{2ad\int \frac{1}{a(c+d)-\frac{a^2d\cos^2(e+fx)}{\sin(e+fx)a+a}}d\frac{a\cos(e+fx)}{\sqrt{\sin(e+fx)a+a}}}{cf} - \frac{2\sqrt{a}\text{arctanh}\left(\frac{\sqrt{a}\cos(e+fx)}{\sqrt{a\sin(e+fx)+a}}\right)}{cf}$$

$$\Big\downarrow \text{221}$$

$$\frac{2\sqrt{a}\sqrt{d}\text{arctanh}\left(\frac{\sqrt{a}\sqrt{d}\cos(e+fx)}{\sqrt{c+d}\sqrt{a\sin(e+fx)+a}}\right)}{cf\sqrt{c+d}} - \frac{2\sqrt{a}\text{arctanh}\left(\frac{\sqrt{a}\cos(e+fx)}{\sqrt{a\sin(e+fx)+a}}\right)}{cf}$$

input
```
Int[(Csc[e + f*x]*Sqrt[a + a*Sin[e + f*x]])/(c + d*Sin[e + f*x]),x]
```

output
```
(-2*Sqrt[a]*ArcTanh[(Sqrt[a]*Cos[e + f*x])/Sqrt[a + a*Sin[e + f*x]]])/(c*f
) + (2*Sqrt[a]*Sqrt[d]*ArcTanh[(Sqrt[a]*Sqrt[d]*Cos[e + f*x])/(Sqrt[c + d]
*Sqrt[a + a*Sin[e + f*x]])])/(c*Sqrt[c + d]*f)
```

3.23.3.1 Defintions of rubi rules used

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 221
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[-a/b, 2]/a)*ArcTanh[x
/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3252
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/((c_.) + (d_.)*sin[(e_.) + (
f_.)*(x_)]), x_Symbol] :> Simp[-2*(b/f)   Subst[Int[1/(b*c + a*d - d*x^2),
x], x, b*(Cos[e + f*x]/Sqrt[a + b*Sin[e + f*x]])], x] /; FreeQ[{a, b, c, d,
 e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]
```

rule 3413
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(sin[(e_.) + (f_.)*(x_)]*((c
_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[1/c   Int[Sqrt[a +
b*Sin[e + f*x]]/Sin[e + f*x], x], x] - Simp[d/c   Int[Sqrt[a + b*Sin[e + f*
x]]/(c + d*Sin[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c
 - a*d, 0] && EqQ[a^2 - b^2, 0]
```

3.23.4 Maple [A] (verified)

Time = 0.58 (sec) , antiderivative size = 120, normalized size of antiderivative = 1.14

method	result	size
default	$\frac{2(1+\sin(fx+e))\sqrt{-a(\sin(fx+e)-1)}\left(d \operatorname{arctanh}\left(\frac{\sqrt{-a(\sin(fx+e)-1)}\,d}{\sqrt{a(c+d)d}}\right)a^{\frac{3}{2}}-\operatorname{arctanh}\left(\frac{\sqrt{-a(\sin(fx+e)-1)}}{\sqrt{a}}\right)a\sqrt{a(c+d)d}\right)}{\sqrt{a}\,c\sqrt{a(c+d)d}\,\cos(fx+e)\sqrt{a+a\sin(fx+e)}\,f}$	120

input
```
int((a+a*sin(f*x+e))^(1/2)/sin(f*x+e)/(c+d*sin(f*x+e)),x,method=_RETURNVER
BOSE)
```

output
```
2/a^(1/2)*(1+sin(f*x+e))*(-a*(sin(f*x+e)-1))^(1/2)*(d*arctanh((-a*(sin(f*x
+e)-1))^(1/2)*d/(a*(c+d)*d)^(1/2))*a^(3/2)-arctanh((-a*(sin(f*x+e)-1))^(1/
2)/a^(1/2))*a*(a*(c+d)*d)^(1/2))/c/(a*(c+d)*d)^(1/2)/cos(f*x+e)/(a+a*sin(f
*x+e))^(1/2)/f
```

3.23.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 242 vs. $2(85) = 170$.

Time = 0.58 (sec) , antiderivative size = 781, normalized size of antiderivative = 7.44

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{c+d\sin(e+fx)}\,dx$$

$$= \left[\frac{\sqrt{\frac{ad}{c+d}}\log\left(\frac{ad^2\cos(fx+e)^3-ac^2-2\,acd-ad^2-(6\,acd+7\,ad^2)\cos(fx+e)^2+4\left((cd+d^2)\cos(fx+e)^2-c^2-4\,cd-3\,d^2-(c^2+3\,cd+2\,d^2)\cos(f\cdots\right.}{d^2\cos(fx+e)^3+(2\,cd+d^2)\cos(fx+\cdots}\right.}{\cdots}\right.$$

input
```
integrate((a+a*sin(f*x+e))^(1/2)/sin(f*x+e)/(c+d*sin(f*x+e)),x, algorithm=
"fricas")
```

output
```
[1/2*(sqrt(a*d/(c + d))*log((a*d^2*cos(f*x + e)^3 - a*c^2 - 2*a*c*d - a*d^
2 - (6*a*c*d + 7*a*d^2)*cos(f*x + e)^2 + 4*((c*d + d^2)*cos(f*x + e)^2 - c
^2 - 4*c*d - 3*d^2 - (c^2 + 3*c*d + 2*d^2)*cos(f*x + e) + (c^2 + 4*c*d + 3
*d^2 + (c*d + d^2)*cos(f*x + e))*sin(f*x + e))*sqrt(a*sin(f*x + e) + a)*sq
rt(a*d/(c + d)) - (a*c^2 + 8*a*c*d + 9*a*d^2)*cos(f*x + e) + (a*d^2*cos(f*
x + e)^2 - a*c^2 - 2*a*c*d - a*d^2 + 2*(3*a*c*d + 4*a*d^2)*cos(f*x + e))*s
in(f*x + e))/(d^2*cos(f*x + e)^3 + (2*c*d + d^2)*cos(f*x + e)^2 - c^2 - 2*
c*d - d^2 - (c^2 + d^2)*cos(f*x + e) + (d^2*cos(f*x + e)^2 - 2*c*d*cos(f*x
 + e) - c^2 - 2*c*d - d^2)*sin(f*x + e))) + sqrt(a)*log((a*cos(f*x + e)^3
- 7*a*cos(f*x + e)^2 - 4*(cos(f*x + e)^2 + (cos(f*x + e) + 3)*sin(f*x + e)
 - 2*cos(f*x + e) - 3)*sqrt(a*sin(f*x + e) + a)*sqrt(a) - 9*a*cos(f*x + e)
 + (a*cos(f*x + e)^2 + 8*a*cos(f*x + e) - a)*sin(f*x + e) - a)/(cos(f*x +
e)^3 + cos(f*x + e)^2 + (cos(f*x + e)^2 - 1)*sin(f*x + e) - cos(f*x + e) -
 1)))/(c*f), 1/2*(2*sqrt(-a*d/(c + d))*arctan(1/2*sqrt(a*sin(f*x + e) + a)
*(d*sin(f*x + e) - c - 2*d)*sqrt(-a*d/(c + d))/(a*d*cos(f*x + e))) + sqrt(
a)*log((a*cos(f*x + e)^3 - 7*a*cos(f*x + e)^2 - 4*(cos(f*x + e)^2 + (cos(f
*x + e) + 3)*sin(f*x + e) - 2*cos(f*x + e) - 3)*sqrt(a*sin(f*x + e) + a)*s
qrt(a) - 9*a*cos(f*x + e) + (a*cos(f*x + e)^2 + 8*a*cos(f*x + e) - a)*sin(
f*x + e) - a)/(cos(f*x + e)^3 + cos(f*x + e)^2 + (cos(f*x + e)^2 - 1)*sin(
f*x + e) - cos(f*x + e) - 1)))/(c*f)]
```

3.23.6 Sympy [F]

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{c+d\sin(e+fx)}\,dx = \int \frac{\sqrt{a\left(\sin\left(e+fx\right)+1\right)}}{\left(c+d\sin\left(e+fx\right)\right)\sin\left(e+fx\right)}\,dx$$

input
```
integrate((a+a*sin(f*x+e))**(1/2)/sin(f*x+e)/(c+d*sin(f*x+e)),x)
```

output
```
Integral(sqrt(a*(sin(e + f*x) + 1))/((c + d*sin(e + f*x))*sin(e + f*x)), x
)
```

3.23.7 Maxima [F]

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{c+d\sin(e+fx)}\,dx=\int \frac{\sqrt{a\sin\left(fx+e\right)+a}}{\left(d\sin\left(fx+e\right)+c\right)\sin\left(fx+e\right)}\,dx$$

input
```
integrate((a+a*sin(f*x+e))^(1/2)/sin(f*x+e)/(c+d*sin(f*x+e)),x, algorithm=
"maxima")
```

output
```
integrate(sqrt(a*sin(f*x + e) + a)/((d*sin(f*x + e) + c)*sin(f*x + e)), x)
```

3.23.8 Giac [A] (verification not implemented)

Time = 0.34 (sec) , antiderivative size = 143, normalized size of antiderivative = 1.36

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{c+d\sin(e+fx)}\,dx$$
$$=\frac{\sqrt{2}\left(\frac{2\sqrt{2}d\arctan\left(\frac{\sqrt{2}d\sin\left(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e\right)}{\sqrt{-cd-d^2}}\right)\operatorname{sgn}\left(\cos\left(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e\right)\right)}{\sqrt{-cd-d^2}c}-\frac{\sqrt{2}\log\left(\frac{\left|-2\sqrt{2}+4\sin\left(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e\right)\right|}{\left|2\sqrt{2}+4\sin\left(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e\right)\right|}\right)\operatorname{sgn}\left(\cos\left(-\frac{1}{4}\pi+\frac{1}{2}fx+\right)\right)}{c}\right)}{2f}$$

input
```
integrate((a+a*sin(f*x+e))^(1/2)/sin(f*x+e)/(c+d*sin(f*x+e)),x, algorithm=
"giac")
```

output
```
1/2*sqrt(2)*(2*sqrt(2)*d*arctan(sqrt(2)*d*sin(-1/4*pi + 1/2*f*x + 1/2*e)/s
qrt(-c*d - d^2))*sgn(cos(-1/4*pi + 1/2*f*x + 1/2*e))/(sqrt(-c*d - d^2)*c)
- sqrt(2)*log(abs(-2*sqrt(2) + 4*sin(-1/4*pi + 1/2*f*x + 1/2*e))/abs(2*sqr
t(2) + 4*sin(-1/4*pi + 1/2*f*x + 1/2*e)))*sgn(cos(-1/4*pi + 1/2*f*x + 1/2*
e))/c)*sqrt(a)/f
```

3.23.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{c+d\sin(e+fx)}\,dx = \int \frac{\sqrt{a+a\,\sin\left(e+f\,x\right)}}{\sin\left(e+f\,x\right)\,\left(c+d\,\sin\left(e+f\,x\right)\right)}\,dx$$

input
```
int((a + a*sin(e + f*x))^(1/2)/(sin(e + f*x)*(c + d*sin(e + f*x))),x)
```

output
```
int((a + a*sin(e + f*x))^(1/2)/(sin(e + f*x)*(c + d*sin(e + f*x))), x)
```

3.24 $\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))}\,dx$

3.24.1 Optimal result

Integrand size = 33, antiderivative size = 165

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))}\,dx = -\frac{2\operatorname{arctanh}\left(\frac{\sqrt{a}\cos(e+fx)}{\sqrt{a+a\sin(e+fx)}}\right)}{\sqrt{a}cf} + \frac{\sqrt{2}\operatorname{arctanh}\left(\frac{\sqrt{a}\cos(e+fx)}{\sqrt{2}\sqrt{a+a\sin(e+fx)}}\right)}{\sqrt{a}(c-d)f} - \frac{2d^{3/2}\operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{d}\cos(e+fx)}{\sqrt{c+d}\sqrt{a+a\sin(e+fx)}}\right)}{\sqrt{a}c(c-d)\sqrt{c+d}f}$$

output
```
-2*arctanh(cos(f*x+e)*a^(1/2)/(a+a*sin(f*x+e))^(1/2))/c/f/a^(1/2)+arctanh(
1/2*cos(f*x+e)*a^(1/2)*2^(1/2)/(a+a*sin(f*x+e))^(1/2))*2^(1/2)/(c-d)/f/a^(
1/2)-2*d^(3/2)*arctanh(cos(f*x+e)*a^(1/2)*d^(1/2)/(c+d)^(1/2)/(a+a*sin(f*x
+e))^(1/2))/c/(c-d)/f/a^(1/2)/(c+d)^(1/2)
```

3.24.2 Mathematica [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.

Time = 3.37 (sec) , antiderivative size = 654, normalized size of antiderivative = 3.96

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))}\,dx$$
$$= \frac{\left(-2\sqrt{c+d}\left((2+2i)(-1)^{3/4}c\,\text{arctanh}\left(\left(\frac{1}{2}+\frac{i}{2}\right)(-1)^{3/4}\left(-1+\tan\left(\frac{1}{4}(e+fx)\right)\right)\right)+(c-d)\left(\log\left(1+\cos\left(\right.\right.\right.\right.\right.}{}$$

input
```
Integrate[Csc[e + f*x]/(Sqrt[a + a*Sin[e + f*x]]*(c + d*Sin[e + f*x])),x]
```

output
```
((-2*Sqrt[c + d]*((2 + 2*I)*(-1)^(3/4)*c*ArcTanh[(1/2 + I/2)*(-1)^(3/4)*(-
1 + Tan[(e + f*x)/4])] + (c - d)*(Log[1 + Cos[(e + f*x)/2] - Sin[(e + f*x)
/2]] - Log[1 - Cos[(e + f*x)/2] + Sin[(e + f*x)/2]])) - d^(3/2)*RootSum[c
+ 4*d*#1 + 2*c*#1^2 - 4*d*#1^3 + c*#1^4 & , (-(d*Log[-#1 + Tan[(e + f*x)/4
]]) + Sqrt[d]*Sqrt[c + d]*Log[-#1 + Tan[(e + f*x)/4]] - c*Log[-#1 + Tan[(e
 + f*x)/4]]*#1 + 2*Sqrt[d]*Sqrt[c + d]*Log[-#1 + Tan[(e + f*x)/4]]*#1 + 3*
d*Log[-#1 + Tan[(e + f*x)/4]]*#1^2 - Sqrt[d]*Sqrt[c + d]*Log[-#1 + Tan[(e
+ f*x)/4]]*#1^2 - c*Log[-#1 + Tan[(e + f*x)/4]]*#1^3)/(-d - c*#1 + 3*d*#1^
2 - c*#1^3) & ] + d^(3/2)*RootSum[c + 4*d*#1 + 2*c*#1^2 - 4*d*#1^3 + c*#1^
4 & , (-(d*Log[-#1 + Tan[(e + f*x)/4]]) - Sqrt[d]*Sqrt[c + d]*Log[-#1 + Ta
n[(e + f*x)/4]] - c*Log[-#1 + Tan[(e + f*x)/4]]*#1 - 2*Sqrt[d]*Sqrt[c + d]
*Log[-#1 + Tan[(e + f*x)/4]]*#1 + 3*d*Log[-#1 + Tan[(e + f*x)/4]]*#1^2 + S
qrt[d]*Sqrt[c + d]*Log[-#1 + Tan[(e + f*x)/4]]*#1^2 - c*Log[-#1 + Tan[(e +
 f*x)/4]]*#1^3)/(-d - c*#1 + 3*d*#1^2 - c*#1^3) & ])*(Cos[(e + f*x)/2] + S
in[(e + f*x)/2]))/(2*c*(c - d)*Sqrt[c + d]*f*Sqrt[a*(1 + Sin[e + f*x])])
```

3.24.3 Rubi [A] (verified)

Time = 1.00 (sec) , antiderivative size = 176, normalized size of antiderivative = 1.07, number of steps used = 12, number of rules used = 11, $\frac{\text{number of rules}}{\text{integrand size}} = 0.333$, Rules used = {3042, 3419, 3042, 3252, 221, 3464, 3042, 3128, 219, 3252, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(e+fx)}{\sqrt{a\sin(e+fx)+a}(c+d\sin(e+fx))}dx$$

$\Big\downarrow$ 3042

$$\int \frac{1}{\sin(e+fx)\sqrt{a\sin(e+fx)+a}(c+d\sin(e+fx))}dx$$

$\Big\downarrow$ 3419

$$\frac{d^2\int \frac{\sqrt{\sin(e+fx)a+a}}{c+d\sin(e+fx)}dx}{ac(c-d)}+\frac{\int \frac{\csc(e+fx)(a(c-d)-ad\sin(e+fx))}{\sqrt{\sin(e+fx)a+a}}dx}{ac(c-d)}$$

$\Big\downarrow$ 3042

$$\frac{d^2\int \frac{\sqrt{\sin(e+fx)a+a}}{c+d\sin(e+fx)}dx}{ac(c-d)}+\frac{\int \frac{a(c-d)-ad\sin(e+fx)}{\sin(e+fx)\sqrt{\sin(e+fx)a+a}}dx}{ac(c-d)}$$

$\Big\downarrow$ 3252

$$\frac{\int \frac{a(c-d)-ad\sin(e+fx)}{\sin(e+fx)\sqrt{\sin(e+fx)a+a}}dx}{ac(c-d)}-\frac{2d^2\int \frac{1}{a(c+d)-\frac{a^2d\cos^2(e+fx)}{\sin(e+fx)a+a}}d\frac{a\cos(e+fx)}{\sqrt{\sin(e+fx)a+a}}}{cf(c-d)}$$

$\Big\downarrow$ 221

$$\frac{\int \frac{a(c-d)-ad\sin(e+fx)}{\sin(e+fx)\sqrt{\sin(e+fx)a+a}}dx}{ac(c-d)}-\frac{2d^{3/2}\text{arctanh}\left(\frac{\sqrt{a}\sqrt{d}\cos(e+fx)}{\sqrt{c+d}\sqrt{a\sin(e+fx)+a}}\right)}{\sqrt{a}cf(c-d)\sqrt{c+d}}$$

$\Big\downarrow$ 3464

$$\frac{(c-d)\int \csc(e+fx)\sqrt{\sin(e+fx)a+a}dx-ac\int \frac{1}{\sqrt{\sin(e+fx)a+a}}dx}{ac(c-d)}-\frac{2d^{3/2}\text{arctanh}\left(\frac{\sqrt{a}\sqrt{d}\cos(e+fx)}{\sqrt{c+d}\sqrt{a\sin(e+fx)+a}}\right)}{\sqrt{a}cf(c-d)\sqrt{c+d}}$$

$\Big\downarrow$ 3042

$$\frac{(c-d)\int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)}dx-ac\int \frac{1}{\sqrt{\sin(e+fx)a+a}}dx}{ac(c-d)}-\frac{2d^{3/2}\text{arctanh}\left(\frac{\sqrt{a}\sqrt{d}\cos(e+fx)}{\sqrt{c+d}\sqrt{a\sin(e+fx)+a}}\right)}{\sqrt{a}cf(c-d)\sqrt{c+d}}$$

$\Big\downarrow$ 3128

$$\frac{\frac{2ac\int \frac{1}{2a-\frac{a^2\cos^2(e+fx)}{\sin(e+fx)a+a}}d\frac{a\cos(e+fx)}{\sqrt{\sin(e+fx)a+a}}}{f}+(c-d)\int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)}dx}{ac(c-d)}-\frac{2d^{3/2}\text{arctanh}\left(\frac{\sqrt{a}\sqrt{d}\cos(e+fx)}{\sqrt{c+d}\sqrt{a\sin(e+fx)+a}}\right)}{\sqrt{a}cf(c-d)\sqrt{c+d}}$$

$\Big\downarrow$ 219

3.24. $\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))}dx$

$$\frac{(c-d)\int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)}dx + \frac{\sqrt{2}\sqrt{a}c\text{arctanh}\left(\frac{\sqrt{a}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}}\right)}{f}}{ac(c-d)} - \frac{2d^{3/2}\text{arctanh}\left(\frac{\sqrt{a}\sqrt{d}\cos(e+fx)}{\sqrt{c+d}\sqrt{a\sin(e+fx)+a}}\right)}{\sqrt{a}cf(c-d)\sqrt{c+d}}$$

$$\Big\downarrow \; 3252$$

$$\frac{\frac{\sqrt{2}\sqrt{a}c\text{arctanh}\left(\frac{\sqrt{a}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}}\right)}{f} - \frac{2a(c-d)\int \frac{1}{a-\frac{a^2\cos^2(e+fx)}{\sin(e+fx)a+a}}d\frac{a\cos(e+fx)}{\sqrt{\sin(e+fx)a+a}}}{f}}{ac(c-d)} - \frac{2d^{3/2}\text{arctanh}\left(\frac{\sqrt{a}\sqrt{d}\cos(e+fx)}{\sqrt{c+d}\sqrt{a\sin(e+fx)+a}}\right)}{\sqrt{a}cf(c-d)\sqrt{c+d}}$$

$$\Big\downarrow \; 219$$

$$\frac{\frac{\sqrt{2}\sqrt{a}c\text{arctanh}\left(\frac{\sqrt{a}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}}\right)}{f} - \frac{2\sqrt{a}(c-d)\text{arctanh}\left(\frac{\sqrt{a}\cos(e+fx)}{\sqrt{a\sin(e+fx)+a}}\right)}{f}}{ac(c-d)} - \frac{2d^{3/2}\text{arctanh}\left(\frac{\sqrt{a}\sqrt{d}\cos(e+fx)}{\sqrt{c+d}\sqrt{a\sin(e+fx)+a}}\right)}{\sqrt{a}cf(c-d)\sqrt{c+d}}$$

input
```
Int[Csc[e + f*x]/(Sqrt[a + a*Sin[e + f*x]]*(c + d*Sin[e + f*x])),x]
```

output
```
((-2*Sqrt[a]*(c - d)*ArcTanh[(Sqrt[a]*Cos[e + f*x])/Sqrt[a + a*Sin[e + f*x
]]])/f + (Sqrt[2]*Sqrt[a]*c*ArcTanh[(Sqrt[a]*Cos[e + f*x])/(Sqrt[2]*Sqrt[a
 + a*Sin[e + f*x]])])/f)/(a*c*(c - d)) - (2*d^(3/2)*ArcTanh[(Sqrt[a]*Sqrt[
d]*Cos[e + f*x])/(Sqrt[c + d]*Sqrt[a + a*Sin[e + f*x]])])/(Sqrt[a]*c*(c -
d)*Sqrt[c + d]*f)
```

3.24.3.1 Defintions of rubi rules used

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 221
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[-a/b, 2]/a)*ArcTanh[x
/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3128
```
Int[1/Sqrt[(a_) + (b_.)*sin[(c_.) + (d_.)*(x_)]], x_Symbol] :> Simp[-2/d
Subst[Int[1/(2*a - x^2), x], x, b*(Cos[c + d*x]/Sqrt[a + b*Sin[c + d*x]])],
 x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0]
```

rule 3252
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/((c_.) + (d_.)*sin[(e_.) + (
f_.)*(x_)]), x_Symbol] :> Simp[-2*(b/f)   Subst[Int[1/(b*c + a*d - d*x^2),
x], x, b*(Cos[e + f*x]/Sqrt[a + b*Sin[e + f*x]])], x] /; FreeQ[{a, b, c, d,
 e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]
```

rule 3419
```
Int[1/(sin[(e_.) + (f_.)*(x_)]*Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]*(
(c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[d^2/(c*(b*c - a*d
))   Int[Sqrt[a + b*Sin[e + f*x]]/(c + d*Sin[e + f*x]), x], x] + Simp[1/(c*
(b*c - a*d))   Int[(b*c - a*d - b*d*Sin[e + f*x])/(Sin[e + f*x]*Sqrt[a + b*
Sin[e + f*x]]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0]
 && EqQ[a^2 - b^2, 0]
```

rule 3464
```
Int[((A_.) + (B_.)*sin[(e_.) + (f_.)*(x_)])/(Sqrt[(a_) + (b_.)*sin[(e_.) +
(f_.)*(x_)]]*((c_.) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[(A
*b - a*B)/(b*c - a*d)   Int[1/Sqrt[a + b*Sin[e + f*x]], x], x] + Simp[(B*c
- A*d)/(b*c - a*d)   Int[Sqrt[a + b*Sin[e + f*x]]/(c + d*Sin[e + f*x]), x],
 x] /; FreeQ[{a, b, c, d, e, f, A, B}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 -
 b^2, 0] && NeQ[c^2 - d^2, 0]
```

3.24.4 Maple [A] (verified)

Time = 0.59 (sec) , antiderivative size = 206, normalized size of antiderivative = 1.25

method	result
default	$\frac{(1+\sin(fx+e))\sqrt{-a(\sin(fx+e)-1)}\left(-2d^2\operatorname{arctanh}\left(\frac{\sqrt{a-a\sin(fx+e)}\,d}{\sqrt{acd+a\,d^2}}\right)a^{\frac{5}{2}}+\sqrt{2}\operatorname{arctanh}\left(\frac{\sqrt{a-a\sin(fx+e)}\,\sqrt{2}}{2\sqrt{a}}\right)a^2c\sqrt{a(c+d)d}-2\,\mathrm{arc}}{(c-d)c\sqrt{a(c+d)d}\,a^{\frac{5}{2}}\cos(fx+e)\sqrt{a+a\sin(fx+e}}$

input
```
int(1/sin(f*x+e)/(c+d*sin(f*x+e))/(a+a*sin(f*x+e))^(1/2),x,method=_RETURNV
ERBOSE)
```

output
```
(1+sin(f*x+e))*(-a*(sin(f*x+e)-1))^(1/2)*(-2*d^2*arctanh((a-a*sin(f*x+e))^
(1/2)*d/(a*c*d+a*d^2)^(1/2))*a^(5/2)+2^(1/2)*arctanh(1/2*(a-a*sin(f*x+e))^
(1/2)*2^(1/2)/a^(1/2))*a^2*c*(a*(c+d)*d)^(1/2)-2*arctanh((a-a*sin(f*x+e))^
(1/2)/a^(1/2))*a^2*(a*(c+d)*d)^(1/2)*c+2*arctanh((a-a*sin(f*x+e))^(1/2)/a^
(1/2))*a^2*(a*(c+d)*d)^(1/2)*d)/(c-d)/c/(a*(c+d)*d)^(1/2)/a^(5/2)/cos(f*x+
e)/(a+a*sin(f*x+e))^(1/2)/f
```

3.24.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 380 vs. $2(136) = 272$.

Time = 1.51 (sec) , antiderivative size = 1044, normalized size of antiderivative = 6.33

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \text{Too large to display}$$

input
```
integrate(1/sin(f*x+e)/(c+d*sin(f*x+e))/(a+a*sin(f*x+e))^(1/2),x, algorith
m="fricas")
```

output
```
[-1/2*(a*d*sqrt(d/(a*c + a*d))*log((d^2*cos(f*x + e)^3 - (6*c*d + 7*d^2)*c
os(f*x + e)^2 - c^2 - 2*c*d - d^2 + 4*((c*d + d^2)*cos(f*x + e)^2 - c^2 -
4*c*d - 3*d^2 - (c^2 + 3*c*d + 2*d^2)*cos(f*x + e) + (c^2 + 4*c*d + 3*d^2
+ (c*d + d^2)*cos(f*x + e))*sin(f*x + e))*sqrt(a*sin(f*x + e) + a)*sqrt(d/
(a*c + a*d)) - (c^2 + 8*c*d + 9*d^2)*cos(f*x + e) + (d^2*cos(f*x + e)^2 -
c^2 - 2*c*d - d^2 + 2*(3*c*d + 4*d^2)*cos(f*x + e))*sin(f*x + e))/(d^2*cos
(f*x + e)^3 + (2*c*d + d^2)*cos(f*x + e)^2 - c^2 - 2*c*d - d^2 - (c^2 + d^
2)*cos(f*x + e) + (d^2*cos(f*x + e)^2 - 2*c*d*cos(f*x + e) - c^2 - 2*c*d -
 d^2)*sin(f*x + e))) + sqrt(2)*sqrt(a)*c*log(-(cos(f*x + e)^2 - (cos(f*x +
 e) - 2)*sin(f*x + e) - 2*sqrt(2)*sqrt(a*sin(f*x + e) + a)*(cos(f*x + e) -
 sin(f*x + e) + 1)/sqrt(a) + 3*cos(f*x + e) + 2)/(cos(f*x + e)^2 - (cos(f*
x + e) + 2)*sin(f*x + e) - cos(f*x + e) - 2)) - sqrt(a)*(c - d)*log((a*cos
(f*x + e)^3 - 7*a*cos(f*x + e)^2 - 4*(cos(f*x + e)^2 + (cos(f*x + e) + 3)*
sin(f*x + e) - 2*cos(f*x + e) - 3)*sqrt(a*sin(f*x + e) + a)*sqrt(a) - 9*a*
cos(f*x + e) + (a*cos(f*x + e)^2 + 8*a*cos(f*x + e) - a)*sin(f*x + e) - a)
/(cos(f*x + e)^3 + cos(f*x + e)^2 + (cos(f*x + e)^2 - 1)*sin(f*x + e) - co
s(f*x + e) - 1)))/((a*c^2 - a*c*d)*f), -1/2*(2*a*d*sqrt(-d/(a*c + a*d))*ar
ctan(1/2*sqrt(a*sin(f*x + e) + a)*(d*sin(f*x + e) - c - 2*d)*sqrt(-d/(a*c
+ a*d))/(d*cos(f*x + e))) + sqrt(2)*sqrt(a)*c*log(-(cos(f*x + e)^2 - (cos(
f*x + e) - 2)*sin(f*x + e) - 2*sqrt(2)*sqrt(a*sin(f*x + e) + a)*(cos(f*...
```

3.24.6 Sympy [F]

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \int \frac{1}{\sqrt{a\left(\sin\left(e+fx\right)+1\right)}\left(c+d\sin\left(e+fx\right)\right)\sin\left(e+fx\right)}\,dx$$

input
```
integrate(1/sin(f*x+e)/(c+d*sin(f*x+e))/(a+a*sin(f*x+e))**(1/2),x)
```

output
```
Integral(1/(sqrt(a*(sin(e + f*x) + 1))*(c + d*sin(e + f*x))*sin(e + f*x)),
 x)
```

3.24.7 Maxima [F]

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \int \frac{1}{\sqrt{a\sin\left(fx+e\right)+a}\left(d\sin\left(fx+e\right)+c\right)\sin\left(fx+e\right)}\,dx$$

input
```
integrate(1/sin(f*x+e)/(c+d*sin(f*x+e))/(a+a*sin(f*x+e))^(1/2),x, algorith
m="maxima")
```

output
```
integrate(1/(sqrt(a*sin(f*x + e) + a)*(d*sin(f*x + e) + c)*sin(f*x + e)),
x)
```

3.24.8 Giac [A] (verification not implemented)

Time = 0.40 (sec) , antiderivative size = 189, normalized size of antiderivative = 1.15

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))}\,dx = -\frac{\sqrt{2}\left(\frac{2\sqrt{2}d^2\arctan\left(\frac{\sqrt{2}d\sin\left(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e\right)}{\sqrt{-cd-d^2}}\right)}{(c^2-cd)\sqrt{-cd-d^2}}+\frac{\sqrt{2}\log\left(\frac{\left|-2\sqrt{2}+4\sin\left(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e\right)\right|}{\left|2\sqrt{2}+4\sin\left(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e\right)\right|}\right)}{c}+\frac{\log(\sin(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e)+1)}{c-d}-\frac{\log(-}{}\right.}{2\sqrt{a}f\operatorname{sgn}\left(\cos\left(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e\right)\right)}$$

input
```
integrate(1/sin(f*x+e)/(c+d*sin(f*x+e))/(a+a*sin(f*x+e))^(1/2),x, algorith
m="giac")
```

output
```
-1/2*sqrt(2)*(2*sqrt(2)*d^2*arctan(sqrt(2)*d*sin(-1/4*pi + 1/2*f*x + 1/2*e
)/sqrt(-c*d - d^2))/((c^2 - c*d)*sqrt(-c*d - d^2)) + sqrt(2)*log(abs(-2*sq
rt(2) + 4*sin(-1/4*pi + 1/2*f*x + 1/2*e))/abs(2*sqrt(2) + 4*sin(-1/4*pi +
1/2*f*x + 1/2*e)))/c + log(sin(-1/4*pi + 1/2*f*x + 1/2*e) + 1)/(c - d) - l
og(-sin(-1/4*pi + 1/2*f*x + 1/2*e) + 1)/(c - d))/(sqrt(a)*f*sgn(cos(-1/4*p
i + 1/2*f*x + 1/2*e)))
```

3.24.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \int \frac{1}{\sin(e+f\,x)\,\sqrt{a+a\,\sin(e+f\,x)}\,(c+d\,\sin(e+f\,x))}\,dx$$

input
```
int(1/(sin(e + f*x)*(a + a*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))),x)
```

output
```
int(1/(sin(e + f*x)*(a + a*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))), x)
```

3.25 $\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}{c+d\sin(e+fx)}\,dx$

3.25.1 Optimal result

Integrand size = 39, antiderivative size = 149

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}{c+d\sin(e+fx)}\,dx = -\frac{2\sqrt{a}\sqrt{g}\arctan\left(\frac{\sqrt{a}\sqrt{g}\cos(e+fx)}{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}\right)}{df} + \frac{2\sqrt{a}\sqrt{c}\sqrt{g}\arctan\left(\frac{\sqrt{a}\sqrt{c}\sqrt{g}\cos(e+fx)}{\sqrt{c+d}\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}\right)}{d\sqrt{c+d}\,f}$$

output
```
-2*arctan(cos(f*x+e)*a^(1/2)*g^(1/2)/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))
^(1/2))*a^(1/2)*g^(1/2)/d/f+2*arctan(cos(f*x+e)*a^(1/2)*c^(1/2)*g^(1/2)/(c
+d)^(1/2)/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2))*a^(1/2)*c^(1/2)*g^(
1/2)/d/f/(c+d)^(1/2)
```

3.25.2 Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 3.40 (sec) , antiderivative size = 662, normalized size of antiderivative = 4.44

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}{c+d\sin(e+fx)}\,dx$$

$$= \frac{\left(\frac{1}{2}+\frac{i}{2}\right)e^{-\frac{5}{2}i(e+fx)}\left(-1+e^{2i(e+fx)}\right)^{5/2}\left(-2i\sqrt{-1+e^{2i(e+fx)}}+\left(i+\frac{c-d}{\sqrt{-c^2+d^2}}\right)\sqrt{-1+e^{2i(e+fx)}}+\left(i+\frac{-c}{\sqrt{-c}}\right.\right.}{}$$

input
```
Integrate[(Sqrt[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]])/(c + d*Sin[e + f
*x]),x]
```

output
```
((1/2 + I/2)*(-1 + E^((2*I)*(e + f*x)))^(5/2)*((-2*I)*Sqrt[-1 + E^((2*I)*(
e + f*x))] + (I + (c - d)/Sqrt[-c^2 + d^2])*Sqrt[-1 + E^((2*I)*(e + f*x))]
 + (I + (-c + d)/Sqrt[-c^2 + d^2])*Sqrt[-1 + E^((2*I)*(e + f*x))] + (2*I)*
ArcTan[Sqrt[-1 + E^((2*I)*(e + f*x))]] + ((I + (-c + d)/Sqrt[-c^2 + d^2])*
(Sqrt[2]*Sqrt[c]*Sqrt[c + I*Sqrt[-c^2 + d^2]]*ArcTan[(d - ((-I)*c + Sqrt[-
c^2 + d^2])*E^(I*(e + f*x)))/(Sqrt[2]*Sqrt[c]*Sqrt[c + I*Sqrt[-c^2 + d^2]]
*Sqrt[-1 + E^((2*I)*(e + f*x))])] + ((-I)*c + Sqrt[-c^2 + d^2])*ArcTanh[E^
(I*(e + f*x))/Sqrt[-1 + E^((2*I)*(e + f*x))]]))/d + ((I + (c - d)/Sqrt[-c^
2 + d^2])*(Sqrt[2]*Sqrt[c]*Sqrt[c - I*Sqrt[-c^2 + d^2]]*ArcTan[(d + (I*c +
 Sqrt[-c^2 + d^2])*E^(I*(e + f*x)))/(Sqrt[2]*Sqrt[c]*Sqrt[c - I*Sqrt[-c^2
+ d^2]]*Sqrt[-1 + E^((2*I)*(e + f*x))])] - (I*c + Sqrt[-c^2 + d^2])*ArcTan
h[E^(I*(e + f*x))/Sqrt[-1 + E^((2*I)*(e + f*x))]]))/d)*Sqrt[g*Sin[e + f*x]
]*Sqrt[a*(1 + Sin[e + f*x])])/(Sqrt[2]*d*E^(((5*I)/2)*(e + f*x))*(((-I)*(-
1 + E^((2*I)*(e + f*x))))/E^(I*(e + f*x)))^(5/2)*f*(Cos[(e + f*x)/2] + Sin
[(e + f*x)/2])*Sqrt[Sin[e + f*x]])
```

3.25.3 Rubi [A] (verified)

Time = 0.85 (sec) , antiderivative size = 149, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}} = 0.179$, Rules used = {3042, 3407, 3042, 3254, 218, 3409, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}{c+d\sin(e+fx)}\,dx$$

$$\Big\downarrow \; 3042$$

$$\int \frac{\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}{c+d\sin(e+fx)}dx$$

$$\Big\downarrow \; 3407$$

$$\frac{g\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}}dx}{d}-\frac{cg\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}dx}{d}$$

$$\Big\downarrow \; 3042$$

$$\frac{g\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}}dx}{d}-\frac{cg\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}dx}{d}$$

$$\Big\downarrow \; 3254$$

$$-\frac{2ag\int \frac{1}{\frac{\cos(e+fx)\cot(e+fx)a^2}{\sin(e+fx)a+a}+a}d\frac{a\cos(e+fx)}{\sqrt{g\sin(e+fx)}\sqrt{\sin(e+fx)a+a}}}{df}-\frac{cg\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}dx}{d}$$

$$\Big\downarrow \; 218$$

$$-\frac{cg\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}dx}{d}-\frac{2\sqrt{a}\sqrt{g}\arctan\left(\frac{\sqrt{a}\sqrt{g}\cos(e+fx)}{\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}\right)}{df}$$

$$\Big\downarrow \; 3409$$

$$\frac{2acg\int \frac{1}{\frac{c\cos(e+fx)\cot(e+fx)a^2}{\sin(e+fx)a+a}+(c+d)a}d\frac{a\cos(e+fx)}{\sqrt{g\sin(e+fx)}\sqrt{\sin(e+fx)a+a}}}{df}-\frac{2\sqrt{a}\sqrt{g}\arctan\left(\frac{\sqrt{a}\sqrt{g}\cos(e+fx)}{\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}\right)}{df}$$

$$\Big\downarrow \; 218$$

$$\frac{2\sqrt{a}\sqrt{c}\sqrt{g}\arctan\left(\frac{\sqrt{a}\sqrt{c}\sqrt{g}\cos(e+fx)}{\sqrt{c+d}\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}\right)}{df\sqrt{c+d}}-\frac{2\sqrt{a}\sqrt{g}\arctan\left(\frac{\sqrt{a}\sqrt{g}\cos(e+fx)}{\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}\right)}{df}$$

input
```
Int[(Sqrt[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]])/(c + d*Sin[e + f*x]),x
]
```

output
```
(-2*Sqrt[a]*Sqrt[g]*ArcTan[(Sqrt[a]*Sqrt[g]*Cos[e + f*x])/(Sqrt[g*Sin[e +
f*x]]*Sqrt[a + a*Sin[e + f*x]])])/(d*f) + (2*Sqrt[a]*Sqrt[c]*Sqrt[g]*ArcTa
n[(Sqrt[a]*Sqrt[c]*Sqrt[g]*Cos[e + f*x])/(Sqrt[c + d]*Sqrt[g*Sin[e + f*x]]
*Sqrt[a + a*Sin[e + f*x]])])/(d*Sqrt[c + d]*f)
```

3.25. $\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}{c+d\sin(e+fx)}dx$

3.25.3.1 Defintions of rubi rules used

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3254
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/Sqrt[(c_.) + (d_.)*sin[(e_.)
 + (f_.)*(x_)]], x_Symbol] :> Simp[-2*(b/f)   Subst[Int[1/(b + d*x^2), x],
x, b*(Cos[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]]))], x
] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0]
 && NeQ[c^2 - d^2, 0]
```

rule 3407
```
Int[(Sqrt[(g_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.
)*(x_)]])/((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] :> Simp[g/d   I
nt[Sqrt[a + b*Sin[e + f*x]]/Sqrt[g*Sin[e + f*x]], x], x] - Simp[c*(g/d)   I
nt[Sqrt[a + b*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c + d*Sin[e + f*x])), x]
, x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && (EqQ[a^2 -
b^2, 0] || EqQ[c^2 - d^2, 0])
```

rule 3409
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(g_.)*sin[(e_.) + (f_.
)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[-2*(b/f
)   Subst[Int[1/(b*c + a*d + c*g*x^2), x], x, b*(Cos[e + f*x]/(Sqrt[g*Sin[e
 + f*x]]*Sqrt[a + b*Sin[e + f*x]]))], x] /; FreeQ[{a, b, c, d, e, f, g}, x]
 && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0]
```

3.25.4 Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 934 vs. $2(117) = 234$.

Time = 3.64 (sec) , antiderivative size = 935, normalized size of antiderivative = 6.28

method	result	size
default	Expression too large to display	935

input
```
int((g*sin(f*x+e))^(1/2)*(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,method=
_RETURNVERBOSE)
```

output
```
1/2/f*(g*sin(f*x+e))^(1/2)*(a*(1+sin(f*x+e)))^(1/2)*(2^(1/2)*(-(c-d)*(c+d)
)^(1/2)*(((-(c-d)*(c+d))^(1/2)+d)*c)^(1/2)*(((-(c-d)*(c+d))^(1/2)-d)*c)^(1
/2)*ln(-(csc(f*x+e)-cot(f*x+e)+(csc(f*x+e)-cot(f*x+e))^(1/2)*2^(1/2)+1)/((
csc(f*x+e)-cot(f*x+e))^(1/2)*2^(1/2)-csc(f*x+e)+cot(f*x+e)-1))+4*2^(1/2)*(
-(c-d)*(c+d))^(1/2)*(((-(c-d)*(c+d))^(1/2)+d)*c)^(1/2)*(((-(c-d)*(c+d))^(1
/2)-d)*c)^(1/2)*arctan((csc(f*x+e)-cot(f*x+e))^(1/2)*2^(1/2)+1)+4*2^(1/2)*
(-(c-d)*(c+d))^(1/2)*(((-(c-d)*(c+d))^(1/2)+d)*c)^(1/2)*(((-(c-d)*(c+d))^(
1/2)-d)*c)^(1/2)*arctan((csc(f*x+e)-cot(f*x+e))^(1/2)*2^(1/2)-1)+2^(1/2)*(
-(c-d)*(c+d))^(1/2)*(((-(c-d)*(c+d))^(1/2)+d)*c)^(1/2)*(((-(c-d)*(c+d))^(1
/2)-d)*c)^(1/2)*ln(-((csc(f*x+e)-cot(f*x+e))^(1/2)*2^(1/2)-csc(f*x+e)+cot(
f*x+e)-1)/(csc(f*x+e)-cot(f*x+e)+(csc(f*x+e)-cot(f*x+e))^(1/2)*2^(1/2)+1))
+4*(-(c-d)*(c+d))^(1/2)*(((-(c-d)*(c+d))^(1/2)+d)*c)^(1/2)*arctanh((csc(f*
x+e)-cot(f*x+e))^(1/2)*c/(((-(c-d)*(c+d))^(1/2)-d)*c)^(1/2))*c-4*(-(c-d)*(
c+d))^(1/2)*(((-(c-d)*(c+d))^(1/2)-d)*c)^(1/2)*arctan((csc(f*x+e)-cot(f*x+
e))^(1/2)*c/(((-(c-d)*(c+d))^(1/2)+d)*c)^(1/2))*c+4*(((-(c-d)*(c+d))^(1/2)
+d)*c)^(1/2)*arctanh((csc(f*x+e)-cot(f*x+e))^(1/2)*c/(((-(c-d)*(c+d))^(1/2
)-d)*c)^(1/2))*c^2-4*(((-(c-d)*(c+d))^(1/2)+d)*c)^(1/2)*arctanh((csc(f*x+e
)-cot(f*x+e))^(1/2)*c/(((-(c-d)*(c+d))^(1/2)-d)*c)^(1/2))*c*d+4*(((-(c-d)*
(c+d))^(1/2)-d)*c)^(1/2)*arctan((csc(f*x+e)-cot(f*x+e))^(1/2)*c/(((-(c-d)*
(c+d))^(1/2)+d)*c)^(1/2))*c^2-4*(((-(c-d)*(c+d))^(1/2)-d)*c)^(1/2)*arct...
```

3.25.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 271 vs. $2(117) = 234$.

Time = 1.47 (sec) , antiderivative size = 3273, normalized size of antiderivative = 21.97

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}{c+d\sin(e+fx)}\,dx = \text{Too large to display}$$

input
```
integrate((g*sin(f*x+e))^(1/2)*(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,
algorithm="fricas")
```

output
```
[1/4*(sqrt(-a*c*g/(c + d))*log(((128*a*c^4 + 256*a*c^3*d + 160*a*c^2*d^2 +
 32*a*c*d^3 + a*d^4)*g*cos(f*x + e)^5 - (128*a*c^4 + 192*a*c^3*d + 64*a*c^
2*d^2 - 4*a*c*d^3 - a*d^4)*g*cos(f*x + e)^4 - 2*(208*a*c^4 + 368*a*c^3*d +
 195*a*c^2*d^2 + 32*a*c*d^3 + a*d^4)*g*cos(f*x + e)^3 + 2*(64*a*c^4 + 94*a
*c^3*d + 29*a*c^2*d^2 - 4*a*c*d^3 - a*d^4)*g*cos(f*x + e)^2 + (289*a*c^4 +
 480*a*c^3*d + 230*a*c^2*d^2 + 32*a*c*d^3 + a*d^4)*g*cos(f*x + e) + 8*((16
*c^4 + 40*c^3*d + 34*c^2*d^2 + 11*c*d^3 + d^4)*cos(f*x + e)^4 + 51*c^4 + 1
10*c^3*d + 76*c^2*d^2 + 18*c*d^3 + d^4 - (24*c^4 + 52*c^3*d + 35*c^2*d^2 +
 7*c*d^3)*cos(f*x + e)^3 - (66*c^4 + 149*c^3*d + 110*c^2*d^2 + 29*c*d^3 +
2*d^4)*cos(f*x + e)^2 + (25*c^4 + 53*c^3*d + 35*c^2*d^2 + 7*c*d^3)*cos(f*x
 + e) - (51*c^4 + 110*c^3*d + 76*c^2*d^2 + 18*c*d^3 + d^4 - (16*c^4 + 40*c
^3*d + 34*c^2*d^2 + 11*c*d^3 + d^4)*cos(f*x + e)^3 - (40*c^4 + 92*c^3*d +
69*c^2*d^2 + 18*c*d^3 + d^4)*cos(f*x + e)^2 + (26*c^4 + 57*c^3*d + 41*c^2*
d^2 + 11*c*d^3 + d^4)*cos(f*x + e))*sin(f*x + e))*sqrt(-a*c*g/(c + d))*sqr
t(a*sin(f*x + e) + a)*sqrt(g*sin(f*x + e)) + (a*c^4 + 4*a*c^3*d + 6*a*c^2*
d^2 + 4*a*c*d^3 + a*d^4)*g + ((128*a*c^4 + 256*a*c^3*d + 160*a*c^2*d^2 + 3
2*a*c*d^3 + a*d^4)*g*cos(f*x + e)^4 + 4*(64*a*c^4 + 112*a*c^3*d + 56*a*c^2
*d^2 + 7*a*c*d^3)*g*cos(f*x + e)^3 - 2*(80*a*c^4 + 144*a*c^3*d + 83*a*c^2*
d^2 + 18*a*c*d^3 + a*d^4)*g*cos(f*x + e)^2 - 4*(72*a*c^4 + 119*a*c^3*d + 5
6*a*c^2*d^2 + 7*a*c*d^3)*g*cos(f*x + e) + (a*c^4 + 4*a*c^3*d + 6*a*c^2*...
```

3.25.6 Sympy [F]

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}{c+d\sin(e+fx)}\,dx=\int \frac{\sqrt{a\left(\sin\left(e+fx\right)+1\right)}\sqrt{g\sin\left(e+fx\right)}}{c+d\sin\left(e+fx\right)}\,dx$$

input
```
integrate((g*sin(f*x+e))**(1/2)*(a+a*sin(f*x+e))**(1/2)/(c+d*sin(f*x+e)),x
)
```

output
```
Integral(sqrt(a*(sin(e + f*x) + 1))*sqrt(g*sin(e + f*x))/(c + d*sin(e + f*
x)), x)
```

3.25.7 Maxima [F]

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}{c+d\sin(e+fx)}\,dx = \int \frac{\sqrt{a\sin\left(fx+e\right)+a}\sqrt{g\sin\left(fx+e\right)}}{d\sin\left(fx+e\right)+c}\,dx$$

input
```
integrate((g*sin(f*x+e))^(1/2)*(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,
algorithm="maxima")
```

output
```
integrate(sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))/(d*sin(f*x + e) +
c), x)
```

3.25.8 Giac [F(-1)]

Timed out.

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}{c+d\sin(e+fx)}\,dx = \text{Timed out}$$

input
```
integrate((g*sin(f*x+e))^(1/2)*(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,
algorithm="giac")
```

output
```
Timed out
```

3.25.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}{c+d\sin(e+fx)}\,dx = \int \frac{\sqrt{g\,\sin\left(e+f\,x\right)}\,\sqrt{a+a\,\sin\left(e+f\,x\right)}}{c+d\,\sin\left(e+f\,x\right)}\,dx$$

input
```
int(((g*sin(e + f*x))^(1/2)*(a + a*sin(e + f*x))^(1/2))/(c + d*sin(e + f*x
)),x)
```

output
```
int(((g*sin(e + f*x))^(1/2)*(a + a*sin(e + f*x))^(1/2))/(c + d*sin(e + f*x
)), x)
```

3.26 $\int \frac{\sqrt{a+a\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}\,dx$

3.26.1 Optimal result

Integrand size = 39, antiderivative size = 83

$$\int \frac{\sqrt{a+a\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}\,dx=-\frac{2\sqrt{a}\arctan\left(\frac{\sqrt{a}\sqrt{c}\sqrt{g}\cos(e+fx)}{\sqrt{c+d}\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}\right)}{\sqrt{c}\sqrt{c+d}f\sqrt{g}}$$

output
```
-2*arctan(cos(f*x+e)*a^(1/2)*c^(1/2)*g^(1/2)/(c+d)^(1/2)/(g*sin(f*x+e))^(1
/2)/(a+a*sin(f*x+e))^(1/2))*a^(1/2)/f/c^(1/2)/(c+d)^(1/2)/g^(1/2)
```

3.26.2 Mathematica [C] (warning: unable to verify)

Result contains complex when optimal does not.

Time = 2.71 (sec) , antiderivative size = 436, normalized size of antiderivative = 5.25

$$\int \frac{\sqrt{a+a\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}\,dx$$
$$=\frac{\left(\frac{1}{4}+\frac{i}{4}\right)g\left(\sqrt{c+i\sqrt{-c^2+d^2}}\left(ic-id+\sqrt{-c^2+d^2}\right)\arctan\left(\frac{d-\left(-ic+\sqrt{-c^2+d^2}\right)(\cos(e+fx)+i\sin(e+fx))}{\sqrt{2}\sqrt{c}\sqrt{c+i\sqrt{-c^2+d^2}}\sqrt{-1+\cos(2(e+fx))+i\sin(2(e+fx))}}\right.\right.}{}$$

input
```
Integrate[Sqrt[a + a*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c + d*Sin[e + f*
x])),x]
```

output
```
((1/4 + I/4)*g*(Sqrt[c + I*Sqrt[-c^2 + d^2]]*(I*c - I*d + Sqrt[-c^2 + d^2]
)*ArcTan[(d - ((-I)*c + Sqrt[-c^2 + d^2])*(Cos[e + f*x] + I*Sin[e + f*x]))
/(Sqrt[2]*Sqrt[c]*Sqrt[c + I*Sqrt[-c^2 + d^2]]*Sqrt[-1 + Cos[2*(e + f*x)]
+ I*Sin[2*(e + f*x)]])] + Sqrt[c - I*Sqrt[-c^2 + d^2]]*((-I)*c + I*d + Sqr
t[-c^2 + d^2])*ArcTan[(d + (I*c + Sqrt[-c^2 + d^2])*(Cos[e + f*x] + I*Sin[
e + f*x]))/(Sqrt[2]*Sqrt[c]*Sqrt[c - I*Sqrt[-c^2 + d^2]]*Sqrt[-1 + Cos[2*(
e + f*x)] + I*Sin[2*(e + f*x)]])])*Sqrt[a*(1 + Sin[e + f*x])]*(Cos[(3*(e +
 f*x))/2] - I*Sin[(3*(e + f*x))/2])*(-1 + Cos[2*(e + f*x)] + I*Sin[2*(e +
f*x)])^(3/2))/(Sqrt[2]*Sqrt[c]*d*Sqrt[-c^2 + d^2]*f*(Cos[(e + f*x)/2] + Si
n[(e + f*x)/2])*(g*Sin[e + f*x])^(3/2))
```

3.26.3 Rubi [A] (verified)

Time = 0.39 (sec) , antiderivative size = 83, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.077$, Rules used = $\{3042, 3409, 218\}$

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sqrt{a\sin(e+fx)+a}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}\,dx$$

$$\Big\downarrow \ 3042$$

$$\int \frac{\sqrt{a\sin(e+fx)+a}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}dx$$

$$\Big\downarrow \ 3409$$

$$-\frac{2a\int \frac{1}{\frac{c\cos(e+fx)\cot(e+fx)a^2}{\sin(e+fx)a+a}+(c+d)a}d\frac{a\cos(e+fx)}{\sqrt{g\sin(e+fx)}\sqrt{\sin(e+fx)a+a}}}{f}$$

$$\Big\downarrow \ 218$$

$$-\frac{2\sqrt{a}\arctan\left(\frac{\sqrt{a}\sqrt{c}\sqrt{g}\cos(e+fx)}{\sqrt{c+d}\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}\right)}{\sqrt{c}f\sqrt{g}\sqrt{c+d}}$$

input
```
Int[Sqrt[a + a*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c + d*Sin[e + f*x])),x
]
```

output
```
(-2*Sqrt[a]*ArcTan[(Sqrt[a]*Sqrt[c]*Sqrt[g]*Cos[e + f*x])/(Sqrt[c + d]*Sqr
t[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]])])/(Sqrt[c]*Sqrt[c + d]*f*Sqrt[
g])
```

3.26.3.1 Defintions of rubi rules used

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3409
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(g_.)*sin[(e_.) + (f_.
)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[-2*(b/f
)   Subst[Int[1/(b*c + a*d + c*g*x^2), x], x, b*(Cos[e + f*x]/(Sqrt[g*Sin[e
 + f*x]]*Sqrt[a + b*Sin[e + f*x]]))], x] /; FreeQ[{a, b, c, d, e, f, g}, x]
 && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0]
```

3.26.4 Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 504 vs. $2(63) = 126$.

Time = 3.17 (sec) , antiderivative size = 505, normalized size of antiderivative = 6.08

method	result
default	$-\frac{2\sqrt{\csc(fx+e)-\cot(fx+e)}\,\sqrt{a(1+\sin(fx+e))}\left(\sqrt{-(c-d)(c+d)}\,\sqrt{\left(\sqrt{-(c-d)(c+d)}+d\right)c}\ \operatorname{arctanh}\left(\frac{\sqrt{\csc(fx+e)-\cot(fx+e)}\,c}{\sqrt{\left(\sqrt{-(c-d)(c+d)}-d\right)c}}\right)+a}{}$

input
```
int((a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))/(g*sin(f*x+e))^(1/2),x,method=
_RETURNVERBOSE)
```

output
```
-2/f*(csc(f*x+e)-cot(f*x+e))^(1/2)*(a*(1+sin(f*x+e)))^(1/2)*((-(c-d)*(c+d)
)^(1/2)*(((-(c-d)*(c+d))^(1/2)+d)*c)^(1/2)*arctanh((csc(f*x+e)-cot(f*x+e))
^(1/2)*c/(((-(c-d)*(c+d))^(1/2)-d)*c)^(1/2))+arctanh((csc(f*x+e)-cot(f*x+e
))^(1/2)*c/(((-(c-d)*(c+d))^(1/2)-d)*c)^(1/2))*(((-(c-d)*(c+d))^(1/2)+d)*c
)^(1/2)*c-arctanh((csc(f*x+e)-cot(f*x+e))^(1/2)*c/(((-(c-d)*(c+d))^(1/2)-d
)*c)^(1/2))*(((-(c-d)*(c+d))^(1/2)+d)*c)^(1/2)*d-(-(c-d)*(c+d))^(1/2)*(((-
(c-d)*(c+d))^(1/2)-d)*c)^(1/2)*arctan((csc(f*x+e)-cot(f*x+e))^(1/2)*c/(((-
(c-d)*(c+d))^(1/2)+d)*c)^(1/2))+arctan((csc(f*x+e)-cot(f*x+e))^(1/2)*c/(((
-(c-d)*(c+d))^(1/2)+d)*c)^(1/2))*(((-(c-d)*(c+d))^(1/2)-d)*c)^(1/2)*c-arct
an((csc(f*x+e)-cot(f*x+e))^(1/2)*c/(((-(c-d)*(c+d))^(1/2)+d)*c)^(1/2))*(((
-(c-d)*(c+d))^(1/2)-d)*c)^(1/2)*d)*(1+cos(f*x+e))/(cos(f*x+e)+sin(f*x+e)+1
)/(g*sin(f*x+e))^(1/2)/(-(c-d)*(c+d))^(1/2)/(((-(c-d)*(c+d))^(1/2)-d)*c)^(
1/2)/(((-(c-d)*(c+d))^(1/2)+d)*c)^(1/2)
```

3.26.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 168 vs. $2(63) = 126$.

Time = 0.89 (sec) , antiderivative size = 1303, normalized size of antiderivative = 15.70

$$\int \frac{\sqrt{a + a\sin(e + fx)}}{\sqrt{g\sin(e + fx)}(c + d\sin(e + fx))}\, dx = \text{Too large to display}$$

input
```
integrate((a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))/(g*sin(f*x+e))^(1/2),x,
algorithm="fricas")
```

output
```
[1/4*sqrt(-a/((c^2 + c*d)*g))*log(((128*a*c^4 + 256*a*c^3*d + 160*a*c^2*d^
2 + 32*a*c*d^3 + a*d^4)*cos(f*x + e)^5 + a*c^4 + 4*a*c^3*d + 6*a*c^2*d^2 +
 4*a*c*d^3 + a*d^4 - (128*a*c^4 + 192*a*c^3*d + 64*a*c^2*d^2 - 4*a*c*d^3 -
 a*d^4)*cos(f*x + e)^4 - 2*(208*a*c^4 + 368*a*c^3*d + 195*a*c^2*d^2 + 32*a
*c*d^3 + a*d^4)*cos(f*x + e)^3 + 2*(64*a*c^4 + 94*a*c^3*d + 29*a*c^2*d^2 -
 4*a*c*d^3 - a*d^4)*cos(f*x + e)^2 - 8*(51*c^5 + 110*c^4*d + 76*c^3*d^2 +
18*c^2*d^3 + c*d^4 + (16*c^5 + 40*c^4*d + 34*c^3*d^2 + 11*c^2*d^3 + c*d^4)
*cos(f*x + e)^4 - (24*c^5 + 52*c^4*d + 35*c^3*d^2 + 7*c^2*d^3)*cos(f*x + e
)^3 - (66*c^5 + 149*c^4*d + 110*c^3*d^2 + 29*c^2*d^3 + 2*c*d^4)*cos(f*x +
e)^2 + (25*c^5 + 53*c^4*d + 35*c^3*d^2 + 7*c^2*d^3)*cos(f*x + e) - (51*c^5
 + 110*c^4*d + 76*c^3*d^2 + 18*c^2*d^3 + c*d^4 - (16*c^5 + 40*c^4*d + 34*c
^3*d^2 + 11*c^2*d^3 + c*d^4)*cos(f*x + e)^3 - (40*c^5 + 92*c^4*d + 69*c^3*
d^2 + 18*c^2*d^3 + c*d^4)*cos(f*x + e)^2 + (26*c^5 + 57*c^4*d + 41*c^3*d^2
 + 11*c^2*d^3 + c*d^4)*cos(f*x + e))*sin(f*x + e))*sqrt(a*sin(f*x + e) + a
)*sqrt(g*sin(f*x + e))*sqrt(-a/((c^2 + c*d)*g)) + (289*a*c^4 + 480*a*c^3*d
 + 230*a*c^2*d^2 + 32*a*c*d^3 + a*d^4)*cos(f*x + e) + (a*c^4 + 4*a*c^3*d +
 6*a*c^2*d^2 + 4*a*c*d^3 + a*d^4 + (128*a*c^4 + 256*a*c^3*d + 160*a*c^2*d^
2 + 32*a*c*d^3 + a*d^4)*cos(f*x + e)^4 + 4*(64*a*c^4 + 112*a*c^3*d + 56*a*
c^2*d^2 + 7*a*c*d^3)*cos(f*x + e)^3 - 2*(80*a*c^4 + 144*a*c^3*d + 83*a*c^2
*d^2 + 18*a*c*d^3 + a*d^4)*cos(f*x + e)^2 - 4*(72*a*c^4 + 119*a*c^3*d +...
```

3.26.6 Sympy [F]

$$\int \frac{\sqrt{a+a\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \int \frac{\sqrt{a\left(\sin\left(e+fx\right)+1\right)}}{\sqrt{g\sin\left(e+fx\right)}\left(c+d\sin\left(e+fx\right)\right)}\,dx$$

input
```
integrate((a+a*sin(f*x+e))**(1/2)/(c+d*sin(f*x+e))/(g*sin(f*x+e))**(1/2),x
)
```

output
```
Integral(sqrt(a*(sin(e + f*x) + 1))/(sqrt(g*sin(e + f*x))*(c + d*sin(e + f
*x))), x)
```

3.26.7 Maxima [F]

$$\int \frac{\sqrt{a+a\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \int \frac{\sqrt{a\sin\left(fx+e\right)+a}}{\left(d\sin\left(fx+e\right)+c\right)\sqrt{g\sin\left(fx+e\right)}}\,dx$$

input
```
integrate((a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))/(g*sin(f*x+e))^(1/2),x,
algorithm="maxima")
```

output
```
integrate(sqrt(a*sin(f*x + e) + a)/((d*sin(f*x + e) + c)*sqrt(g*sin(f*x +
e))), x)
```

3.26.8 Giac [F]

$$\int \frac{\sqrt{a+a\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \int \frac{\sqrt{a\sin\left(fx+e\right)+a}}{\left(d\sin\left(fx+e\right)+c\right)\sqrt{g\sin\left(fx+e\right)}}\,dx$$

input
```
integrate((a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))/(g*sin(f*x+e))^(1/2),x,
algorithm="giac")
```

output
```
integrate(sqrt(a*sin(f*x + e) + a)/((d*sin(f*x + e) + c)*sqrt(g*sin(f*x +
e))), x)
```

3.26.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{a+a\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \int \frac{\sqrt{a+a\,\sin\left(e+f\,x\right)}}{\sqrt{g\,\sin\left(e+f\,x\right)}\,\left(c+d\,\sin\left(e+f\,x\right)\right)}\,dx$$

input
```
int((a + a*sin(e + f*x))^(1/2)/((g*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x)
)),x)
```

output
```
int((a + a*sin(e + f*x))^(1/2)/((g*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x)
)), x)
```

3.27 $\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))}\,dx$

3.27.1 Optimal result

Integrand size = 39, antiderivative size = 166

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \frac{\sqrt{2}\sqrt{g}\arctan\left(\frac{\sqrt{a}\sqrt{g}\cos(e+fx)}{\sqrt{2}\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}\right)}{\sqrt{a}(c-d)f} - \frac{2\sqrt{c}\sqrt{g}\arctan\left(\frac{\sqrt{a}\sqrt{c}\sqrt{g}\cos(e+fx)}{\sqrt{c+d}\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}\right)}{\sqrt{a}(c-d)\sqrt{c+d}f}$$

output
```
arctan(1/2*cos(f*x+e)*a^(1/2)*g^(1/2)*2^(1/2)/(g*sin(f*x+e))^(1/2)/(a+a*si
n(f*x+e))^(1/2))*2^(1/2)*g^(1/2)/(c-d)/f/a^(1/2)-2*arctan(cos(f*x+e)*a^(1/
2)*c^(1/2)*g^(1/2)/(c+d)^(1/2)/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2)
)*c^(1/2)*g^(1/2)/(c-d)/f/a^(1/2)/(c+d)^(1/2)
```

3.27.2 Mathematica [A] (verified)

Time = 5.90 (sec) , antiderivative size = 280, normalized size of antiderivative = 1.69

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))}\,dx = -\frac{\left(2\sqrt{c}\sqrt{-c^2+d^2}\arctan\left(\sqrt{\tan\left(\frac{1}{2}(e+fx)\right)}\right)-\sqrt{d-\sqrt{-c^2+d^2}}\left(-c+d+\sqrt{-c^2+d^2}\right)\arctan\left(\frac{\sqrt{c}\sqrt{\cdots}}{\sqrt{d\cdots}}\right.\right.}{\sqrt{c}(c-d)\sqrt{-c^2+d^2}f\cdots}$$

input
```
Integrate[Sqrt[g*Sin[e + f*x]]/(Sqrt[a + a*Sin[e + f*x]]*(c + d*Sin[e + f*
x])),x]
```

output
```
-(((2*Sqrt[c]*Sqrt[-c^2 + d^2]*ArcTan[Sqrt[Tan[(e + f*x)/2]]] - Sqrt[d - S
qrt[-c^2 + d^2]]*(-c + d + Sqrt[-c^2 + d^2])*ArcTan[(Sqrt[c]*Sqrt[Tan[(e +
 f*x)/2]])/Sqrt[d - Sqrt[-c^2 + d^2]]] - (c - d + Sqrt[-c^2 + d^2])*Sqrt[d
 + Sqrt[-c^2 + d^2]]*ArcTan[(Sqrt[c]*Sqrt[Tan[(e + f*x)/2]])/Sqrt[d + Sqrt
[-c^2 + d^2]]])*Sqrt[g*Sin[e + f*x]]*(1 + Tan[(e + f*x)/2]))/(Sqrt[c]*(c -
 d)*Sqrt[-c^2 + d^2]*f*Sqrt[a*(1 + Sin[e + f*x])]*Sqrt[Tan[(e + f*x)/2]]))
```

3.27.3 Rubi [A] (verified)

Time = 0.90 (sec) , antiderivative size = 166, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}} = 0.179$, Rules used = {3042, 3415, 3042, 3261, 218, 3409, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a\sin(e+fx)+a}(c+d\sin(e+fx))}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a\sin(e+fx)+a}(c+d\sin(e+fx))}dx$$

$$\Big\downarrow \text{3415}$$

$$\frac{cg\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}dx}{a(c-d)}-\frac{g\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{\sin(e+fx)a+a}}dx}{c-d}$$

$$\Big\downarrow \text{3042}$$

$$\frac{cg\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}dx}{a(c-d)}-\frac{g\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{\sin(e+fx)a+a}}dx}{c-d}$$

$$\Big\downarrow \text{3261}$$

$$\frac{2ag\int \frac{1}{\frac{\cos(e+fx)\cot(e+fx)a^3}{\sin(e+fx)a+a}+2a^2}d\frac{a\cos(e+fx)}{\sqrt{g\sin(e+fx)}\sqrt{\sin(e+fx)a+a}}}{f(c-d)}+\frac{cg\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}dx}{a(c-d)}$$

$$\Big\downarrow \text{218}$$

$$\frac{cg\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}dx}{a(c-d)}+\frac{\sqrt{2}\sqrt{g}\arctan\left(\frac{\sqrt{a}\sqrt{g}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}\right)}{\sqrt{a}f(c-d)}$$

$$\Big\downarrow \text{3409}$$

$$\frac{\sqrt{2}\sqrt{g}\arctan\left(\frac{\sqrt{a}\sqrt{g}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}\right)}{\sqrt{a}f(c-d)}-\frac{2cg\int \frac{1}{\frac{c\cos(e+fx)\cot(e+fx)a^2}{\sin(e+fx)a+a}+(c+d)a}d\frac{a\cos(e+fx)}{\sqrt{g\sin(e+fx)}\sqrt{\sin(e+fx)a+a}}}{f(c-d)}$$

$$\Big\downarrow \text{218}$$

$$\frac{\sqrt{2}\sqrt{g}\arctan\left(\frac{\sqrt{a}\sqrt{g}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}\right)}{\sqrt{a}f(c-d)}-\frac{2\sqrt{c}\sqrt{g}\arctan\left(\frac{\sqrt{a}\sqrt{c}\sqrt{g}\cos(e+fx)}{\sqrt{c+d}\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}\right)}{\sqrt{a}f(c-d)\sqrt{c+d}}$$

input
```
Int[Sqrt[g*Sin[e + f*x]]/(Sqrt[a + a*Sin[e + f*x]]*(c + d*Sin[e + f*x])),x
]
```

output
```
(Sqrt[2]*Sqrt[g]*ArcTan[(Sqrt[a]*Sqrt[g]*Cos[e + f*x])/(Sqrt[2]*Sqrt[g*Sin
[e + f*x]]*Sqrt[a + a*Sin[e + f*x]])])/(Sqrt[a]*(c - d)*f) - (2*Sqrt[c]*Sq
rt[g]*ArcTan[(Sqrt[a]*Sqrt[c]*Sqrt[g]*Cos[e + f*x])/(Sqrt[c + d]*Sqrt[g*Si
n[e + f*x]]*Sqrt[a + a*Sin[e + f*x]])])/(Sqrt[a]*(c - d)*Sqrt[c + d]*f)
```

3.27.3.1 Defintions of rubi rules used

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3261
```
Int[1/(Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(c_.) + (d_.)*sin[(e
_.) + (f_.)*(x_)]]), x_Symbol] :> Simp[-2*(a/f)   Subst[Int[1/(2*b^2 - (a*c
 - b*d)*x^2), x], x, b*(Cos[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*S
in[e + f*x]]))], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] &&
 EqQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]
```

rule 3409
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(g_.)*sin[(e_.) + (f_.
)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[-2*(b/f
)   Subst[Int[1/(b*c + a*d + c*g*x^2), x], x, b*(Cos[e + f*x]/(Sqrt[g*Sin[e
 + f*x]]*Sqrt[a + b*Sin[e + f*x]]))], x] /; FreeQ[{a, b, c, d, e, f, g}, x]
 && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0]
```

rule 3415
```
Int[Sqrt[(g_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.
)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[(-a)*(g
/(b*c - a*d))   Int[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]), x],
x] + Simp[c*(g/(b*c - a*d))   Int[Sqrt[a + b*Sin[e + f*x]]/(Sqrt[g*Sin[e +
f*x]]*(c + d*Sin[e + f*x])), x], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && N
eQ[b*c - a*d, 0] && (EqQ[a^2 - b^2, 0] || EqQ[c^2 - d^2, 0])
```

3.27.4 Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 588 vs. $2(131) = 262$.

Time = 3.14 (sec) , antiderivative size = 589, normalized size of antiderivative = 3.55

method	result
default	$\frac{\sqrt{g\sin(fx+e)}\left(\sqrt{-(c-d)(c+d)}\sqrt{\left(\sqrt{-(c-d)(c+d)}-d\right)c}\arctan\left(\frac{\sqrt{\csc(fx+e)-\cot(fx+e)}\,c}{\sqrt{\left(\sqrt{-(c-d)(c+d)}+d\right)c}}\right)c-\sqrt{\left(\sqrt{-(c-d)(c+d)}-d\right)c}\arctan\left(\right.\cdots}{\cdots}$

input
```
int((g*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))/(a+a*sin(f*x+e))^(1/2),x,method=
_RETURNVERBOSE)
```

output
```
1/f*(g*sin(f*x+e))^(1/2)*((-(c-d)*(c+d))^(1/2)*(((-(c-d)*(c+d))^(1/2)-d)*c
)^(1/2)*arctan((csc(f*x+e)-cot(f*x+e))^(1/2)*c/(((-(c-d)*(c+d))^(1/2)+d)*c
)^(1/2))*c-(((-(c-d)*(c+d))^(1/2)-d)*c)^(1/2)*arctan((csc(f*x+e)-cot(f*x+e
))^(1/2)*c/(((-(c-d)*(c+d))^(1/2)+d)*c)^(1/2))*c^2+(((-(c-d)*(c+d))^(1/2)-
d)*c)^(1/2)*arctan((csc(f*x+e)-cot(f*x+e))^(1/2)*c/(((-(c-d)*(c+d))^(1/2)+
d)*c)^(1/2))*c*d-(-(c-d)*(c+d))^(1/2)*(((-(c-d)*(c+d))^(1/2)+d)*c)^(1/2)*a
rctanh((csc(f*x+e)-cot(f*x+e))^(1/2)*c/(((-(c-d)*(c+d))^(1/2)-d)*c)^(1/2))
*c-(((-(c-d)*(c+d))^(1/2)+d)*c)^(1/2)*arctanh((csc(f*x+e)-cot(f*x+e))^(1/2
)*c/(((-(c-d)*(c+d))^(1/2)-d)*c)^(1/2))*c^2+(((-(c-d)*(c+d))^(1/2)+d)*c)^(
1/2)*arctanh((csc(f*x+e)-cot(f*x+e))^(1/2)*c/(((-(c-d)*(c+d))^(1/2)-d)*c)^
(1/2))*c*d-2*arctan((csc(f*x+e)-cot(f*x+e))^(1/2))*(-(c-d)*(c+d))^(1/2)*((
(-(c-d)*(c+d))^(1/2)-d)*c)^(1/2)*(((-(c-d)*(c+d))^(1/2)+d)*c)^(1/2))*(cos(
f*x+e)+sin(f*x+e)+1)/(1+cos(f*x+e))/(a*(1+sin(f*x+e)))^(1/2)/(csc(f*x+e)-c
ot(f*x+e))^(1/2)/(c-d)/(-(c-d)*(c+d))^(1/2)/(((-(c-d)*(c+d))^(1/2)-d)*c)^(
1/2)/(((-(c-d)*(c+d))^(1/2)+d)*c)^(1/2)
```

3.27.5 Fricas [A] (verification not implemented)

Time = 1.33 (sec) , antiderivative size = 3048, normalized size of antiderivative = 18.36

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \text{Too large to display}$$

input
```
integrate((g*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))/(a+a*sin(f*x+e))^(1/2),x,
algorithm="fricas")
```

output
```
[-1/4*(sqrt(2)*sqrt(-g/a)*log((17*g*cos(f*x + e)^3 - 4*sqrt(2)*(3*cos(f*x
+ e)^2 + (3*cos(f*x + e) + 4)*sin(f*x + e) - cos(f*x + e) - 4)*sqrt(a*sin(
f*x + e) + a)*sqrt(g*sin(f*x + e))*sqrt(-g/a) + 3*g*cos(f*x + e)^2 - 18*g*
cos(f*x + e) + (17*g*cos(f*x + e)^2 + 14*g*cos(f*x + e) - 4*g)*sin(f*x + e
) - 4*g)/(cos(f*x + e)^3 + 3*cos(f*x + e)^2 + (cos(f*x + e)^2 - 2*cos(f*x
+ e) - 4)*sin(f*x + e) - 2*cos(f*x + e) - 4)) + sqrt(-c*g/(a*c + a*d))*log
(((128*c^4 + 256*c^3*d + 160*c^2*d^2 + 32*c*d^3 + d^4)*g*cos(f*x + e)^5 -
(128*c^4 + 192*c^3*d + 64*c^2*d^2 - 4*c*d^3 - d^4)*g*cos(f*x + e)^4 - 2*(2
08*c^4 + 368*c^3*d + 195*c^2*d^2 + 32*c*d^3 + d^4)*g*cos(f*x + e)^3 + 2*(6
4*c^4 + 94*c^3*d + 29*c^2*d^2 - 4*c*d^3 - d^4)*g*cos(f*x + e)^2 + (289*c^4
 + 480*c^3*d + 230*c^2*d^2 + 32*c*d^3 + d^4)*g*cos(f*x + e) + 8*((16*c^4 +
 40*c^3*d + 34*c^2*d^2 + 11*c*d^3 + d^4)*cos(f*x + e)^4 + 51*c^4 + 110*c^3
*d + 76*c^2*d^2 + 18*c*d^3 + d^4 - (24*c^4 + 52*c^3*d + 35*c^2*d^2 + 7*c*d
^3)*cos(f*x + e)^3 - (66*c^4 + 149*c^3*d + 110*c^2*d^2 + 29*c*d^3 + 2*d^4)
*cos(f*x + e)^2 + (25*c^4 + 53*c^3*d + 35*c^2*d^2 + 7*c*d^3)*cos(f*x + e)
- (51*c^4 + 110*c^3*d + 76*c^2*d^2 + 18*c*d^3 + d^4 - (16*c^4 + 40*c^3*d +
 34*c^2*d^2 + 11*c*d^3 + d^4)*cos(f*x + e)^3 - (40*c^4 + 92*c^3*d + 69*c^2
*d^2 + 18*c*d^3 + d^4)*cos(f*x + e)^2 + (26*c^4 + 57*c^3*d + 41*c^2*d^2 +
11*c*d^3 + d^4)*cos(f*x + e))*sin(f*x + e))*sqrt(a*sin(f*x + e) + a)*sqrt(
g*sin(f*x + e))*sqrt(-c*g/(a*c + a*d)) + (c^4 + 4*c^3*d + 6*c^2*d^2 + 4...
```

3.27.6 Sympy [F]

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))}\,dx$$
$$=\int \frac{\sqrt{g\sin\left(e+fx\right)}}{\sqrt{a\left(\sin\left(e+fx\right)+1\right)}\left(c+d\sin\left(e+fx\right)\right)}\,dx$$

input
```
integrate((g*sin(f*x+e))**(1/2)/(c+d*sin(f*x+e))/(a+a*sin(f*x+e))**(1/2),x
)
```

output
```
Integral(sqrt(g*sin(e + f*x))/(sqrt(a*(sin(e + f*x) + 1))*(c + d*sin(e + f
*x))), x)
```

3.27.7 Maxima [F]

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))}\,dx=\int \frac{\sqrt{g\sin\left(fx+e\right)}}{\sqrt{a\sin\left(fx+e\right)+a}(d\sin\left(fx+e\right)+c)}\,dx$$

input
```
integrate((g*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))/(a+a*sin(f*x+e))^(1/2),x,
algorithm="maxima")
```

output
```
integrate(sqrt(g*sin(f*x + e))/(sqrt(a*sin(f*x + e) + a)*(d*sin(f*x + e) +
 c)), x)
```

3.27.8 Giac [F(-1)]

Timed out.

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))}\,dx=\text{Timed out}$$

input
```
integrate((g*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))/(a+a*sin(f*x+e))^(1/2),x,
algorithm="giac")
```

output
```
Timed out
```

3.27.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))}\,dx$$
$$=\int \frac{\sqrt{g\,\sin\left(e+f\,x\right)}}{\sqrt{a+a\,\sin\left(e+f\,x\right)}\,\left(c+d\,\sin\left(e+f\,x\right)\right)}\,dx$$

input
```
int((g*sin(e + f*x))^(1/2)/((a + a*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x)
)),x)
```

output
```
int((g*sin(e + f*x))^(1/2)/((a + a*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x)
)), x)
```

3.28 $\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))}\,dx$

3.28.1 Optimal result

Integrand size = 39, antiderivative size = 168

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))}\,dx = -\frac{\sqrt{2}\arctan\left(\frac{\sqrt{a}\sqrt{g}\cos(e+fx)}{\sqrt{2}\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}\right)}{\sqrt{a}(c-d)f\sqrt{g}}+\frac{2d\arctan\left(\frac{\sqrt{a}\sqrt{c}\sqrt{g}\cos(e+fx)}{\sqrt{c+d}\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}}\right)}{\sqrt{a}\sqrt{c}(c-d)\sqrt{c+d}f\sqrt{g}}$$

output
```
-arctan(1/2*cos(f*x+e)*a^(1/2)*g^(1/2)*2^(1/2)/(g*sin(f*x+e))^(1/2)/(a+a*s
in(f*x+e))^(1/2))*2^(1/2)/(c-d)/f/a^(1/2)/g^(1/2)+2*d*arctan(cos(f*x+e)*a^
(1/2)*c^(1/2)*g^(1/2)/(c+d)^(1/2)/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1
/2))/(c-d)/f/a^(1/2)/c^(1/2)/(c+d)^(1/2)/g^(1/2)
```

3.28.2 Mathematica [A] (verified)

Time = 8.22 (sec) , antiderivative size = 289, normalized size of antiderivative = 1.72

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))}\,dx = -\frac{2\left(-2\arctan\left(\sqrt{\tan\left(\frac{1}{2}(e+fx)\right)}\right)+\frac{d\left(1+\frac{c-d}{\sqrt{-c^2+d^2}}\right)\arctan\left(\frac{\sqrt{c}\sqrt{\tan\left(\frac{1}{2}(e+fx)\right)}}{\sqrt{d-\sqrt{-c^2+d^2}}}\right)}{\sqrt{c}\sqrt{d-\sqrt{-c^2+d^2}}}+\frac{d\left(-c+d+\sqrt{-c^2+d^2}\right)\arctan\left(\frac{\sqrt{c}\sqrt{\text{ta}}}{\sqrt{d-}}\right.}{\sqrt{c}\sqrt{-c^2+d^2}\sqrt{d+\sqrt{-c^2+}}}\right.}{(c-d)f\sqrt{g\sin(e+fx)}\sqrt{a(1+}}$$

input
```
Integrate[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]]*(c + d*Sin[e +
f*x])),x]
```

output
```
(-2*(-2*ArcTan[Sqrt[Tan[(e + f*x)/2]]] + (d*(1 + (c - d)/Sqrt[-c^2 + d^2])
*ArcTan[(Sqrt[c]*Sqrt[Tan[(e + f*x)/2]])/Sqrt[d - Sqrt[-c^2 + d^2]]])/(Sqr
t[c]*Sqrt[d - Sqrt[-c^2 + d^2]]) + (d*(-c + d + Sqrt[-c^2 + d^2])*ArcTan[(
Sqrt[c]*Sqrt[Tan[(e + f*x)/2]])/Sqrt[d + Sqrt[-c^2 + d^2]]])/(Sqrt[c]*Sqrt
[-c^2 + d^2]*Sqrt[d + Sqrt[-c^2 + d^2]]))*Cos[(e + f*x)/2]*(Cos[(e + f*x)/
2] + Sin[(e + f*x)/2])*Sqrt[Tan[(e + f*x)/2]])/((c - d)*f*Sqrt[g*Sin[e + f
*x]]*Sqrt[a*(1 + Sin[e + f*x])])
```

3.28.3 Rubi [A] (verified)

Time = 0.89 (sec) , antiderivative size = 168, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}} = 0.179$, Rules used = {3042, 3417, 3042, 3261, 218, 3409, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{1}{\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{1}{\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}dx$$

$$\Big\downarrow \text{3417}$$

$$\frac{\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{\sin(e+fx)a+a}}dx}{c-d}-\frac{d\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}dx}{a(c-d)}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{\sin(e+fx)a+a}}dx}{c-d}-\frac{d\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}dx}{a(c-d)}$$

$$\Big\downarrow \text{3261}$$

$$-\frac{2a\int \frac{1}{\frac{\cos(e+fx)\cot(e+fx)a^3}{\sin(e+fx)a+a}+2a^2}d\frac{a\cos(e+fx)}{\sqrt{g\sin(e+fx)}\sqrt{\sin(e+fx)a+a}}}{f(c-d)}-\frac{d\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}dx}{a(c-d)}$$

$$\Big\downarrow \text{218}$$

$$-\frac{d\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}dx}{a(c-d)}-\frac{\sqrt{2}\arctan\left(\frac{\sqrt{a}\sqrt{g}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}\right)}{\sqrt{a}f\sqrt{g}(c-d)}$$

$\Big\downarrow$ 3409

$$\frac{2d\int \frac{1}{\frac{c\cos(e+fx)\cot(e+fx)a^2}{\sin(e+fx)a+a}+(c+d)a}d\frac{a\cos(e+fx)}{\sqrt{g\sin(e+fx)}\sqrt{\sin(e+fx)a+a}}}{f(c-d)}-\frac{\sqrt{2}\arctan\left(\frac{\sqrt{a}\sqrt{g}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}\right)}{\sqrt{a}f\sqrt{g}(c-d)}$$

$\Big\downarrow$ 218

$$\frac{2d\arctan\left(\frac{\sqrt{a}\sqrt{c}\sqrt{g}\cos(e+fx)}{\sqrt{c+d}\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}\right)}{\sqrt{a}\sqrt{c}f\sqrt{g}(c-d)\sqrt{c+d}}-\frac{\sqrt{2}\arctan\left(\frac{\sqrt{a}\sqrt{g}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}\sqrt{g\sin(e+fx)}}\right)}{\sqrt{a}f\sqrt{g}(c-d)}$$

input
```
Int[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]]*(c + d*Sin[e + f*x]))
,x]
```

output
```
-((Sqrt[2]*ArcTan[(Sqrt[a]*Sqrt[g]*Cos[e + f*x])/(Sqrt[2]*Sqrt[g*Sin[e + f
*x]]*Sqrt[a + a*Sin[e + f*x]])])/(Sqrt[a]*(c - d)*f*Sqrt[g])) + (2*d*ArcTa
n[(Sqrt[a]*Sqrt[c]*Sqrt[g]*Cos[e + f*x])/(Sqrt[c + d]*Sqrt[g*Sin[e + f*x]]
*Sqrt[a + a*Sin[e + f*x]])])/(Sqrt[a]*Sqrt[c]*(c - d)*Sqrt[c + d]*f*Sqrt[g
])
```

3.28.3.1 Defintions of rubi rules used

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3261
```
Int[1/(Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(c_.) + (d_.)*sin[(e
_.) + (f_.)*(x_)]]), x_Symbol] :> Simp[-2*(a/f)   Subst[Int[1/(2*b^2 - (a*c
 - b*d)*x^2), x], x, b*(Cos[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*S
in[e + f*x]]))], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] &&
 EqQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]
```

rule 3409
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(g_.)*sin[(e_.) + (f_.
)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[-2*(b/f
)   Subst[Int[1/(b*c + a*d + c*g*x^2), x], x, b*(Cos[e + f*x]/(Sqrt[g*Sin[e
 + f*x]]*Sqrt[a + b*Sin[e + f*x]]))], x] /; FreeQ[{a, b, c, d, e, f, g}, x]
 && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0]
```

rule 3417
```
Int[1/(Sqrt[(g_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(a_) + (b_.)*sin[(e_.) + (f
_.)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[b/(b*
c - a*d)   Int[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]), x], x] -
Simp[d/(b*c - a*d)   Int[Sqrt[a + b*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c
+ d*Sin[e + f*x])), x], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c -
a*d, 0] && (EqQ[a^2 - b^2, 0] || EqQ[c^2 - d^2, 0])
```

3.28.4 Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 579 vs. $2(133) = 266$.

Time = 3.49 (sec) , antiderivative size = 580, normalized size of antiderivative = 3.45

method	result
default	$-\frac{\sqrt{\csc(fx+e)-\cot(fx+e)}\left(\sqrt{-(c-d)(c+d)}\sqrt{\left(\sqrt{-(c-d)(c+d)}-d\right)c}\,\arctan\left(\frac{\sqrt{\csc(fx+e)-\cot(fx+e)}\,c}{\sqrt{\left(\sqrt{-(c-d)(c+d)}+d\right)c}}\right)d-\sqrt{\left(\sqrt{-(c-d)(c+d)}-\right.}}{}$

input
```
int(1/(c+d*sin(f*x+e))/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2),x,metho
d=_RETURNVERBOSE)
```

output
```
-1/f*(csc(f*x+e)-cot(f*x+e))^(1/2)*((-(c-d)*(c+d))^(1/2)*(((-(c-d)*(c+d))^
(1/2)-d)*c)^(1/2)*arctan((csc(f*x+e)-cot(f*x+e))^(1/2)*c/(((-(c-d)*(c+d))^
(1/2)+d)*c)^(1/2))*d-(((-(c-d)*(c+d))^(1/2)-d)*c)^(1/2)*arctan((csc(f*x+e)
-cot(f*x+e))^(1/2)*c/(((-(c-d)*(c+d))^(1/2)+d)*c)^(1/2))*c*d+(((-(c-d)*(c+
d))^(1/2)-d)*c)^(1/2)*arctan((csc(f*x+e)-cot(f*x+e))^(1/2)*c/(((-(c-d)*(c+
d))^(1/2)+d)*c)^(1/2))*d^2-(-(c-d)*(c+d))^(1/2)*(((-(c-d)*(c+d))^(1/2)+d)*
c)^(1/2)*arctanh((csc(f*x+e)-cot(f*x+e))^(1/2)*c/(((-(c-d)*(c+d))^(1/2)-d)
*c)^(1/2))*d-(((-(c-d)*(c+d))^(1/2)+d)*c)^(1/2)*arctanh((csc(f*x+e)-cot(f*
x+e))^(1/2)*c/(((-(c-d)*(c+d))^(1/2)-d)*c)^(1/2))*c*d+(((-(c-d)*(c+d))^(1/
2)+d)*c)^(1/2)*arctanh((csc(f*x+e)-cot(f*x+e))^(1/2)*c/(((-(c-d)*(c+d))^(1
/2)-d)*c)^(1/2))*d^2-2*arctan((csc(f*x+e)-cot(f*x+e))^(1/2))*(-(c-d)*(c+d)
)^(1/2)*(((-(c-d)*(c+d))^(1/2)-d)*c)^(1/2)*(((-(c-d)*(c+d))^(1/2)+d)*c)^(1
/2))*(cos(f*x+e)+sin(f*x+e)+1)/(a*(1+sin(f*x+e)))^(1/2)/(g*sin(f*x+e))^(1/
2)/(c-d)/(-(c-d)*(c+d))^(1/2)/(((-(c-d)*(c+d))^(1/2)-d)*c)^(1/2)/(((-(c-d)
*(c+d))^(1/2)+d)*c)^(1/2)
```

3.28.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 310 vs. $2(133) = 266$.

Time = 1.58 (sec) , antiderivative size = 3175, normalized size of antiderivative = 18.90

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \text{Too large to display}$$

input
```
integrate(1/(c+d*sin(f*x+e))/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2),x
, algorithm="fricas")
```

output
```
[-1/4*(sqrt(2)*(a*c^2 + a*c*d)*g*sqrt(-1/(a*g))*log((4*sqrt(2)*(3*cos(f*x
+ e)^2 + (3*cos(f*x + e) + 4)*sin(f*x + e) - cos(f*x + e) - 4)*sqrt(a*sin(
f*x + e) + a)*sqrt(g*sin(f*x + e))*sqrt(-1/(a*g)) + 17*cos(f*x + e)^3 + 3*
cos(f*x + e)^2 + (17*cos(f*x + e)^2 + 14*cos(f*x + e) - 4)*sin(f*x + e) -
18*cos(f*x + e) - 4)/(cos(f*x + e)^3 + 3*cos(f*x + e)^2 + (cos(f*x + e)^2
- 2*cos(f*x + e) - 4)*sin(f*x + e) - 2*cos(f*x + e) - 4)) - sqrt(-(a*c^2 +
 a*c*d)*g)*d*log(((128*a*c^4 + 256*a*c^3*d + 160*a*c^2*d^2 + 32*a*c*d^3 +
a*d^4)*g*cos(f*x + e)^5 - (128*a*c^4 + 192*a*c^3*d + 64*a*c^2*d^2 - 4*a*c*
d^3 - a*d^4)*g*cos(f*x + e)^4 - 2*(208*a*c^4 + 368*a*c^3*d + 195*a*c^2*d^2
 + 32*a*c*d^3 + a*d^4)*g*cos(f*x + e)^3 + 2*(64*a*c^4 + 94*a*c^3*d + 29*a*
c^2*d^2 - 4*a*c*d^3 - a*d^4)*g*cos(f*x + e)^2 + (289*a*c^4 + 480*a*c^3*d +
 230*a*c^2*d^2 + 32*a*c*d^3 + a*d^4)*g*cos(f*x + e) + 8*((16*c^3 + 24*c^2*
d + 10*c*d^2 + d^3)*cos(f*x + e)^4 - (24*c^3 + 28*c^2*d + 7*c*d^2)*cos(f*x
 + e)^3 + 51*c^3 + 59*c^2*d + 17*c*d^2 + d^3 - (66*c^3 + 83*c^2*d + 27*c*d
^2 + 2*d^3)*cos(f*x + e)^2 + (25*c^3 + 28*c^2*d + 7*c*d^2)*cos(f*x + e) +
((16*c^3 + 24*c^2*d + 10*c*d^2 + d^3)*cos(f*x + e)^3 - 51*c^3 - 59*c^2*d -
 17*c*d^2 - d^3 + (40*c^3 + 52*c^2*d + 17*c*d^2 + d^3)*cos(f*x + e)^2 - (2
6*c^3 + 31*c^2*d + 10*c*d^2 + d^3)*cos(f*x + e))*sin(f*x + e))*sqrt(-(a*c^
2 + a*c*d)*g)*sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*x + e)) + (a*c^4 + 4*a
*c^3*d + 6*a*c^2*d^2 + 4*a*c*d^3 + a*d^4)*g + ((128*a*c^4 + 256*a*c^3*d...
```

3.28.6 Sympy [F]

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \int \frac{1}{\sqrt{a\left(\sin\left(e+fx\right)+1\right)}\sqrt{g\sin\left(e+fx\right)}\left(c+d\sin\left(e+fx\right)\right)}\,dx$$

input
```
integrate(1/(c+d*sin(f*x+e))/(g*sin(f*x+e))**(1/2)/(a+a*sin(f*x+e))**(1/2)
,x)
```

output
```
Integral(1/(sqrt(a*(sin(e + f*x) + 1))*sqrt(g*sin(e + f*x))*(c + d*sin(e +
 f*x))), x)
```

3.28.7 Maxima [F]

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \int \frac{1}{\sqrt{a\sin\left(fx+e\right)+a}(d\sin\left(fx+e\right)+c)\sqrt{g\sin\left(fx+e\right)}}\,dx$$

input
```
integrate(1/(c+d*sin(f*x+e))/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2),x
, algorithm="maxima")
```

output
```
integrate(1/(sqrt(a*sin(f*x + e) + a)*(d*sin(f*x + e) + c)*sqrt(g*sin(f*x
+ e))), x)
```

3.28.8 Giac [F(-2)]

Exception generated.

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \text{Exception raised: TypeError}$$

input
```
integrate(1/(c+d*sin(f*x+e))/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2),x
, algorithm="giac")
```

output
```
Exception raised: TypeError >> an error occurred running a Giac command:IN
PUT:sage2:=int(sage0,sageVARx):;OUTPUT:index.cc index_m operator + Error:
Bad Argument Valueindex.cc index_m operator + Error: Bad Argument ValueDon
e
```

3.28.9 Mupad [F(-1)]

Timed out.

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \int \frac{1}{\sqrt{g\,\sin\left(e+f\,x\right)}\,\sqrt{a+a\,\sin\left(e+f\,x\right)}\,\left(c+d\,\sin\left(e+f\,x\right)\right)}\,dx$$

input
```
int(1/((g*sin(e + f*x))^(1/2)*(a + a*sin(e + f*x))^(1/2)*(c + d*sin(e + f*
x))),x)
```

output
```
int(1/((g*sin(e + f*x))^(1/2)*(a + a*sin(e + f*x))^(1/2)*(c + d*sin(e + f*
x))), x)
```

3.28. $\int \frac{1}{\sqrt{g \sin(e+fx)}\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))}\,dx$

3.29 $\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{c+c\sin(e+fx)}\,dx$

3.29.1 Optimal result

Integrand size = 33, antiderivative size = 238

$$\begin{aligned}\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{c+c\sin(e+fx)}\,dx &= \frac{E\left(\frac{1}{2}\left(e-\frac{\pi}{2}+fx\right)\middle|\frac{2b}{a+b}\right)\sqrt{a+b\sin(e+fx)}}{cf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}} \\ &\quad - \frac{(a-b)\,\text{EllipticF}\left(\frac{1}{2}\left(e-\frac{\pi}{2}+fx\right),\frac{2b}{a+b}\right)\sqrt{\frac{a+b\sin(e+fx)}{a+b}}}{cf\sqrt{a+b\sin(e+fx)}} \\ &\quad + \frac{2a\,\text{EllipticPi}\left(2,\frac{1}{2}\left(e-\frac{\pi}{2}+fx\right),\frac{2b}{a+b}\right)\sqrt{\frac{a+b\sin(e+fx)}{a+b}}}{cf\sqrt{a+b\sin(e+fx)}} + \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(c+c\sin(e+fx))}\end{aligned}$$

output
```
cos(f*x+e)*(a+b*sin(f*x+e))^(1/2)/f/(c+c*sin(f*x+e))-(sin(1/2*e+1/4*Pi+1/2
*f*x)^2)^(1/2)/sin(1/2*e+1/4*Pi+1/2*f*x)*EllipticE(cos(1/2*e+1/4*Pi+1/2*f*
x),2^(1/2)*(b/(a+b))^(1/2))*(a+b*sin(f*x+e))^(1/2)/c/f/((a+b*sin(f*x+e))/(
a+b))^(1/2)+(a-b)*(sin(1/2*e+1/4*Pi+1/2*f*x)^2)^(1/2)/sin(1/2*e+1/4*Pi+1/2
*f*x)*EllipticF(cos(1/2*e+1/4*Pi+1/2*f*x),2^(1/2)*(b/(a+b))^(1/2))*((a+b*s
in(f*x+e))/(a+b))^(1/2)/c/f/(a+b*sin(f*x+e))^(1/2)-2*a*(sin(1/2*e+1/4*Pi+1
/2*f*x)^2)^(1/2)/sin(1/2*e+1/4*Pi+1/2*f*x)*EllipticPi(cos(1/2*e+1/4*Pi+1/2
*f*x),2,2^(1/2)*(b/(a+b))^(1/2))*((a+b*sin(f*x+e))/(a+b))^(1/2)/c/f/(a+b*s
in(f*x+e))^(1/2)
```

3.29.2 Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 19.05 (sec) , antiderivative size = 452, normalized size of antiderivative = 1.90

$$\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{c+c\sin(e+fx)}\,dx$$
$$= \frac{\left(\cos\left(\frac{1}{2}(e+fx)\right)+\sin\left(\frac{1}{2}(e+fx)\right)\right)\left(-8\sin\left(\frac{1}{2}(e+fx)\right)\sqrt{a+b\sin(e+fx)}+\left(\cos\left(\frac{1}{2}(e+fx)\right)+\sin\left(\frac{1}{2}\right.\right.\right.}{}$$

input
```
Integrate[(Csc[e + f*x]*Sqrt[a + b*Sin[e + f*x]])/(c + c*Sin[e + f*x]),x]
```

output
```
((Cos[(e + f*x)/2] + Sin[(e + f*x)/2])*(-8*Sin[(e + f*x)/2]*Sqrt[a + b*Sin
[e + f*x]] + (Cos[(e + f*x)/2] + Sin[(e + f*x)/2])*(((-2*I)*(-2*a*(a - b)*
EllipticE[I*ArcSinh[Sqrt[-(a + b)^(-1)]*Sqrt[a + b*Sin[e + f*x]]], (a + b)
/(a - b)] + b*(-2*a*EllipticF[I*ArcSinh[Sqrt[-(a + b)^(-1)]*Sqrt[a + b*Sin
[e + f*x]]], (a + b)/(a - b)] + b*EllipticPi[(a + b)/a, I*ArcSinh[Sqrt[-(a
 + b)^(-1)]*Sqrt[a + b*Sin[e + f*x]]], (a + b)/(a - b)]))*Sec[e + f*x]*Sqr
t[-((b*(-1 + Sin[e + f*x]))/(a + b))]*Sqrt[-((b*(1 + Sin[e + f*x]))/(a - b
))])/(a*b*Sqrt[-(a + b)^(-1)]) + 4*Sqrt[a + b*Sin[e + f*x]] - (4*b*Ellipti
cF[(-2*e + Pi - 2*f*x)/4, (2*b)/(a + b)]*Sqrt[(a + b*Sin[e + f*x])/(a + b)
])/Sqrt[a + b*Sin[e + f*x]] - (2*(4*a + b)*EllipticPi[2, (-2*e + Pi - 2*f*
x)/4, (2*b)/(a + b)]*Sqrt[(a + b*Sin[e + f*x])/(a + b)])/Sqrt[a + b*Sin[e
+ f*x]])))/(4*c*f*(1 + Sin[e + f*x]))
```

3.29.3 Rubi [A] (verified)

Time = 1.77 (sec) , antiderivative size = 273, normalized size of antiderivative = 1.15, number of steps used = 17, number of rules used = 17, $\frac{\text{number of rules}}{\text{integrand size}} = 0.515$, Rules used = {3042, 3414, 3042, 3247, 27, 3042, 3231, 3042, 3134, 3042, 3132, 3142, 3042, 3140, 3286, 3042, 3284}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{c\sin(e+fx)+c}\,dx$$
$$\Big\downarrow \text{3042}$$

$$\int \frac{\sqrt{a+b\sin(e+fx)}}{\sin(e+fx)(c\sin(e+fx)+c)}dx$$

$$\Big\downarrow \text{3414}$$

$$\frac{a\int \frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}}dx}{c}-(a-b)\int \frac{1}{\sqrt{a+b\sin(e+fx)}(\sin(e+fx)c+c)}dx$$

$$\Big\downarrow \text{3042}$$

$$\frac{a\int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}}dx}{c}-(a-b)\int \frac{1}{\sqrt{a+b\sin(e+fx)}(\sin(e+fx)c+c)}dx$$

$$\Big\downarrow \text{3247}$$

$$\frac{a\int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}}dx}{c}-(a-b)\left(\frac{b\int -\frac{\sin(e+fx)c+c}{2\sqrt{a+b\sin(e+fx)}}dx}{c^2(a-b)}-\frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}\right)$$

$$\Big\downarrow \text{27}$$

$$\frac{a\int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}}dx}{c}-(a-b)\left(-\frac{b\int \frac{\sin(e+fx)c+c}{\sqrt{a+b\sin(e+fx)}}dx}{2c^2(a-b)}-\frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}\right)$$

$$\Big\downarrow \text{3042}$$

$$\frac{a\int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}}dx}{c}-(a-b)\left(-\frac{b\int \frac{\sin(e+fx)c+c}{\sqrt{a+b\sin(e+fx)}}dx}{2c^2(a-b)}-\frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}\right)$$

$$\Big\downarrow \text{3231}$$

$$\frac{a\int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}}dx}{c}-(a-b)\left(-\frac{b\left(\frac{c\int \sqrt{a+b\sin(e+fx)}dx}{b}-\frac{c(a-b)\int \frac{1}{\sqrt{a+b\sin(e+fx)}}dx}{b}\right)}{2c^2(a-b)}-\frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}\right)$$

$$\Big\downarrow \text{3042}$$

$$\frac{a\int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}}dx}{c}-(a-b)\left(-\frac{b\left(\frac{c\int \sqrt{a+b\sin(e+fx)}dx}{b}-\frac{c(a-b)\int \frac{1}{\sqrt{a+b\sin(e+fx)}}dx}{b}\right)}{2c^2(a-b)}-\frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}\right)$$

$$\Big\downarrow \text{3134}$$

3.29. $\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{c+c\sin(e+fx)}\,dx$

$$\frac{a\int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}}dx}{c}-(a-$$

$$b)\left(-\frac{b\left(\frac{c\sqrt{a+b\sin(e+fx)}\int \sqrt{\frac{a}{a+b}+\frac{b\sin(e+fx)}{a+b}}dx}{b\sqrt{\frac{a+b\sin(e+fx)}{a+b}}}-\frac{c(a-b)\int \frac{1}{\sqrt{a+b\sin(e+fx)}}dx}{b}\right)}{2c^2(a-b)}-\frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}\right)$$

$\Big\downarrow$ 3042

$$\frac{a\int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}}dx}{c}-(a-$$

$$b)\left(-\frac{b\left(\frac{c\sqrt{a+b\sin(e+fx)}\int \sqrt{\frac{a}{a+b}+\frac{b\sin(e+fx)}{a+b}}dx}{b\sqrt{\frac{a+b\sin(e+fx)}{a+b}}}-\frac{c(a-b)\int \frac{1}{\sqrt{a+b\sin(e+fx)}}dx}{b}\right)}{2c^2(a-b)}-\frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}\right)$$

$\Big\downarrow$ 3132

$$\frac{a\int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}}dx}{c}-(a-$$

$$b)\left(-\frac{b\left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right)|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}}-\frac{c(a-b)\int \frac{1}{\sqrt{a+b\sin(e+fx)}}dx}{b}\right)}{2c^2(a-b)}-\frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}\right)$$

$\Big\downarrow$ 3142

$$\frac{a\int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}}dx}{c}-(a-$$

$$b)\left(-\frac{b\left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right)|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}}-\frac{c(a-b)\sqrt{\frac{a+b\sin(e+fx)}{a+b}}\int \frac{1}{\sqrt{\frac{a}{a+b}+\frac{b\sin(e+fx)}{a+b}}}dx}{b\sqrt{a+b\sin(e+fx)}}\right)}{2c^2(a-b)}-\frac{\cos(e+fx)\sqrt{a+b\sin(e-}{f(a-b)(c\sin(e+fx)-}\right.$$

$\Big\downarrow$ 3042

3.29. $\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{c+c\sin(e+fx)}\,dx$

$$b)\left(-\frac{b\left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right)|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}}-\frac{c(a-b)\sqrt{\frac{a+b\sin(e+fx)}{a+b}}\int\frac{1}{\sqrt{\frac{a}{a+b}+\frac{b\sin(e+fx)}{a+b}}}dx}{b\sqrt{a+b\sin(e+fx)}}\right)}{2c^2(a-b)}-\frac{\cos(e+fx)\sqrt{a+b\sin(e-}{f(a-b)(c\sin(e+fx)-}\right.$$

$$\frac{a\int\frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}}dx}{c}-(a-$$

$\downarrow$ 3140

$$\frac{a\int\frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}}dx}{c}-(a-$$

$$b)\left(-\frac{b\left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right)|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}}-\frac{2c(a-b)\sqrt{\frac{a+b\sin(e+fx)}{a+b}}\text{EllipticF}\left(\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right),\frac{2b}{a+b}\right)}{bf\sqrt{a+b\sin(e+fx)}}\right)}{2c^2(a-b)}-\frac{\cos(e+fx)\sqrt{a+bs}{f(a-b)(c\sin(e+}\right.$$

$\downarrow$ 3286

$$\frac{a\sqrt{\frac{a+b\sin(e+fx)}{a+b}}\int\frac{\csc(e+fx)}{\sqrt{\frac{a}{a+b}+\frac{b\sin(e+fx)}{a+b}}}dx}{c\sqrt{a+b\sin(e+fx)}}-(a-$$

$$b)\left(-\frac{b\left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right)|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}}-\frac{2c(a-b)\sqrt{\frac{a+b\sin(e+fx)}{a+b}}\text{EllipticF}\left(\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right),\frac{2b}{a+b}\right)}{bf\sqrt{a+b\sin(e+fx)}}\right)}{2c^2(a-b)}-\frac{\cos(e+fx)\sqrt{a+bs}{f(a-b)(c\sin(e+}\right.$$

$\downarrow$ 3042

$$\frac{a\sqrt{\frac{a+b\sin(e+fx)}{a+b}}\int\frac{1}{\sin(e+fx)\sqrt{\frac{a}{a+b}+\frac{b\sin(e+fx)}{a+b}}}dx}{c\sqrt{a+b\sin(e+fx)}}-(a-$$

$$b)\left(-\frac{b\left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right)|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}}-\frac{2c(a-b)\sqrt{\frac{a+b\sin(e+fx)}{a+b}}\text{EllipticF}\left(\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right),\frac{2b}{a+b}\right)}{bf\sqrt{a+b\sin(e+fx)}}\right)}{2c^2(a-b)}-\frac{\cos(e+fx)\sqrt{a+bs}{f(a-b)(c\sin(e+}\right.$$

$\downarrow$ 3284

3.29. $\int\frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{c+c\sin(e+fx)}dx$

$$\frac{2a\sqrt{\frac{a+b\sin(e+fx)}{a+b}}\,\text{EllipticPi}\left(2,\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right),\frac{2b}{a+b}\right)}{cf\sqrt{a+b\sin(e+fx)}}-(a-$$

$$b)\left(-\frac{b\left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right)|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}}-\frac{2c(a-b)\sqrt{\frac{a+b\sin(e+fx)}{a+b}}\,\text{EllipticF}\left(\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right),\frac{2b}{a+b}\right)}{bf\sqrt{a+b\sin(e+fx)}}\right)}{2c^2(a-b)}-\frac{\cos(e+fx)\sqrt{a+b\sin}}{f(a-b)(c\sin(e+}\right.$$

input
```
Int[(Csc[e + f*x]*Sqrt[a + b*Sin[e + f*x]])/(c + c*Sin[e + f*x]),x]
```

output
```
(2*a*EllipticPi[2, (e - Pi/2 + f*x)/2, (2*b)/(a + b)]*Sqrt[(a + b*Sin[e +
f*x])/(a + b)])/(c*f*Sqrt[a + b*Sin[e + f*x]]) - (a - b)*(-((Cos[e + f*x]*
Sqrt[a + b*Sin[e + f*x]])/((a - b)*f*(c + c*Sin[e + f*x]))) - (b*((2*c*Ell
ipticE[(e - Pi/2 + f*x)/2, (2*b)/(a + b)]*Sqrt[a + b*Sin[e + f*x]])/(b*f*S
qrt[(a + b*Sin[e + f*x])/(a + b)]) - (2*(a - b)*c*EllipticF[(e - Pi/2 + f*
x)/2, (2*b)/(a + b)]*Sqrt[(a + b*Sin[e + f*x])/(a + b)])/(b*f*Sqrt[a + b*S
in[e + f*x]])))/(2*(a - b)*c^2))
```

3.29.3.1 Defintions of rubi rules used

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3132
```
Int[Sqrt[(a_) + (b_.)*sin[(c_.) + (d_.)*(x_)]], x_Symbol] :> Simp[2*(Sqrt[a
 + b]/d)*EllipticE[(1/2)*(c - Pi/2 + d*x), 2*(b/(a + b))], x] /; FreeQ[{a,
b, c, d}, x] && NeQ[a^2 - b^2, 0] && GtQ[a + b, 0]
```

rule 3134
```
Int[Sqrt[(a_) + (b_.)*sin[(c_.) + (d_.)*(x_)]], x_Symbol] :> Simp[Sqrt[a +
b*Sin[c + d*x]]/Sqrt[(a + b*Sin[c + d*x])/(a + b)]   Int[Sqrt[a/(a + b) + (
b/(a + b))*Sin[c + d*x]], x], x] /; FreeQ[{a, b, c, d}, x] && NeQ[a^2 - b^2
, 0] &&  !GtQ[a + b, 0]
```

rule 3140
```
Int[1/Sqrt[(a_) + (b_.)*sin[(c_.) + (d_.)*(x_)]], x_Symbol] :> Simp[(2/(d*S
qrt[a + b]))*EllipticF[(1/2)*(c - Pi/2 + d*x), 2*(b/(a + b))], x] /; FreeQ[
{a, b, c, d}, x] && NeQ[a^2 - b^2, 0] && GtQ[a + b, 0]
```

rule 3142
```
Int[1/Sqrt[(a_) + (b_.)*sin[(c_.) + (d_.)*(x_)]], x_Symbol] :> Simp[Sqrt[(a
 + b*Sin[c + d*x])/(a + b)]/Sqrt[a + b*Sin[c + d*x]]   Int[1/Sqrt[a/(a + b)
 + (b/(a + b))*Sin[c + d*x]], x], x] /; FreeQ[{a, b, c, d}, x] && NeQ[a^2 -
 b^2, 0] &&  !GtQ[a + b, 0]
```

rule 3231
```
Int[((c_.) + (d_.)*sin[(e_.) + (f_.)*(x_)])/Sqrt[(a_) + (b_.)*sin[(e_.) + (
f_.)*(x_)]], x_Symbol] :> Simp[(b*c - a*d)/b   Int[1/Sqrt[a + b*Sin[e + f*x
]], x], x] + Simp[d/b   Int[Sqrt[a + b*Sin[e + f*x]], x], x] /; FreeQ[{a, b
, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0]
```

rule 3247
```
Int[((c_.) + (d_.)*sin[(e_.) + (f_.)*(x_)])^(n_)/((a_) + (b_.)*sin[(e_.) +
(f_.)*(x_)]), x_Symbol] :> Simp[(-b^2)*Cos[e + f*x]*((c + d*Sin[e + f*x])^(
n + 1)/(a*f*(b*c - a*d)*(a + b*Sin[e + f*x]))), x] + Simp[d/(a*(b*c - a*d))
   Int[(c + d*Sin[e + f*x])^n*(a*n - b*(n + 1)*Sin[e + f*x]), x], x] /; Fre
eQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0] && NeQ[
c^2 - d^2, 0] && LtQ[n, 0] && (IntegerQ[2*n] || EqQ[c, 0])
```

rule 3284
```
Int[1/(((a_.) + (b_.)*sin[(e_.) + (f_.)*(x_)])*Sqrt[(c_.) + (d_.)*sin[(e_.)
 + (f_.)*(x_)]]), x_Symbol] :> Simp[(2/(f*(a + b)*Sqrt[c + d]))*EllipticPi[
2*(b/(a + b)), (1/2)*(e - Pi/2 + f*x), 2*(d/(c + d))], x] /; FreeQ[{a, b, c
, d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2,
0] && GtQ[c + d, 0]
```

rule 3286
```
Int[1/(((a_.) + (b_.)*sin[(e_.) + (f_.)*(x_)])*Sqrt[(c_.) + (d_.)*sin[(e_.)
 + (f_.)*(x_)]]), x_Symbol] :> Simp[Sqrt[(c + d*Sin[e + f*x])/(c + d)]/Sqrt
[c + d*Sin[e + f*x]]   Int[1/((a + b*Sin[e + f*x])*Sqrt[c/(c + d) + (d/(c +
 d))*Sin[e + f*x]]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*
d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0] &&  !GtQ[c + d, 0]
```

3.29. $\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{c+c\sin(e+fx)}\,dx$

rule 3414
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(sin[(e_.) + (f_.)*(x_)]*((c
_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[a/c   Int[1/(Sin[e
+ f*x]*Sqrt[a + b*Sin[e + f*x]]), x], x] + Simp[(b*c - a*d)/c   Int[1/(Sqrt
[a + b*Sin[e + f*x]]*(c + d*Sin[e + f*x])), x], x] /; FreeQ[{a, b, c, d, e,
 f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0]
```

3.29.4 Maple [A] (verified)

Time = 1.32 (sec) , antiderivative size = 593, normalized size of antiderivative = 2.49

method	result
default	$\sqrt{-\left(-b\sin\left(fx+e\right)-a\right)\left(\cos^{2}\left(fx+e\right)\right)}\left(-\frac{2\left(\frac{a}{b}-1\right)\sqrt{\frac{a+b\sin\left(fx+e\right)}{a-b}}\,\sqrt{\frac{\left(1-\sin\left(fx+e\right)\right)b}{a+b}}\,\sqrt{\frac{\left(-\sin\left(fx+e\right)-1\right)b}{a-b}}\,b\Pi\left(\sqrt{\frac{a+b\sin\left(fx+e\right)}{a-b}},-\frac{\left(-\frac{a}{b}+1\right)b}{a}\right.}{\sqrt{-\left(-b\sin\left(fx+e\right)-a\right)\left(\cos^{2}\left(fx+e\right)\right)}}\right.$

input
```
int((a+b*sin(f*x+e))^(1/2)/sin(f*x+e)/(c+c*sin(f*x+e)),x,method=_RETURNVER
BOSE)
```

output
```
(-(-b*sin(f*x+e)-a)*cos(f*x+e)^2)^(1/2)/c*(-2*(1/b*a-1)*((a+b*sin(f*x+e))/
(a-b))^(1/2)*(1/(a+b)*(1-sin(f*x+e))*b)^(1/2)*(1/(a-b)*(-sin(f*x+e)-1)*b)^
(1/2)/(-(-b*sin(f*x+e)-a)*cos(f*x+e)^2)^(1/2)*b*EllipticPi(((a+b*sin(f*x+e
))/(a-b))^(1/2),-(-1/b*a+1)*b/a,((a-b)/(a+b))^(1/2))+(-a+b)*(-(-b*sin(f*x+
e)^2-a*sin(f*x+e)+b*sin(f*x+e)+a)/(a-b)/((1+sin(f*x+e))*(sin(f*x+e)-1)*(-b
*sin(f*x+e)-a))^(1/2)-2*b/(2*a-2*b)*(1/b*a-1)*((a+b*sin(f*x+e))/(a-b))^(1/
2)*(1/(a+b)*(1-sin(f*x+e))*b)^(1/2)*(1/(a-b)*(-sin(f*x+e)-1)*b)^(1/2)/(-(-
b*sin(f*x+e)-a)*cos(f*x+e)^2)^(1/2)*EllipticF(((a+b*sin(f*x+e))/(a-b))^(1/
2),((a-b)/(a+b))^(1/2))-b/(a-b)*(1/b*a-1)*((a+b*sin(f*x+e))/(a-b))^(1/2)*(
1/(a+b)*(1-sin(f*x+e))*b)^(1/2)*(1/(a-b)*(-sin(f*x+e)-1)*b)^(1/2)/(-(-b*si
n(f*x+e)-a)*cos(f*x+e)^2)^(1/2)*((-1/b*a-1)*EllipticE(((a+b*sin(f*x+e))/(a
-b))^(1/2),((a-b)/(a+b))^(1/2))+EllipticF(((a+b*sin(f*x+e))/(a-b))^(1/2),(
(a-b)/(a+b))^(1/2)))))/cos(f*x+e)/(a+b*sin(f*x+e))^(1/2)/f
```

3.29.5 Fricas [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{c+c\sin(e+fx)}\,dx = \text{Timed out}$$

input
```
integrate((a+b*sin(f*x+e))^(1/2)/sin(f*x+e)/(c+c*sin(f*x+e)),x, algorithm=
"fricas")
```

output
```
Timed out
```

3.29.6 Sympy [F]

$$\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{c+c\sin(e+fx)}\,dx = \frac{\int \frac{\sqrt{a+b\sin(e+fx)}}{\sin^2(e+fx)+\sin(e+fx)}\,dx}{c}$$

input
```
integrate((a+b*sin(f*x+e))**(1/2)/sin(f*x+e)/(c+c*sin(f*x+e)),x)
```

output
```
Integral(sqrt(a + b*sin(e + f*x))/(sin(e + f*x)**2 + sin(e + f*x)), x)/c
```

3.29.7 Maxima [F]

$$\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{c+c\sin(e+fx)}\,dx = \int \frac{\sqrt{b\sin(fx+e)+a}}{(c\sin(fx+e)+c)\sin(fx+e)}\,dx$$

input
```
integrate((a+b*sin(f*x+e))^(1/2)/sin(f*x+e)/(c+c*sin(f*x+e)),x, algorithm=
"maxima")
```

output
```
integrate(sqrt(b*sin(f*x + e) + a)/((c*sin(f*x + e) + c)*sin(f*x + e)), x)
```

3.29.8 Giac [F]

$$\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{c+c\sin(e+fx)}\,dx=\int \frac{\sqrt{b\sin(fx+e)+a}}{(c\sin(fx+e)+c)\sin(fx+e)}\,dx$$

input
```
integrate((a+b*sin(f*x+e))^(1/2)/sin(f*x+e)/(c+c*sin(f*x+e)),x, algorithm=
"giac")
```

output
```
integrate(sqrt(b*sin(f*x + e) + a)/((c*sin(f*x + e) + c)*sin(f*x + e)), x)
```

3.29.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{c+c\sin(e+fx)}\,dx=\int \frac{\sqrt{a+b\,\sin(e+f\,x)}}{\sin(e+f\,x)\,\left(c+c\,\sin(e+f\,x)\right)}\,dx$$

input
```
int((a + b*sin(e + f*x))^(1/2)/(sin(e + f*x)*(c + c*sin(e + f*x))),x)
```

output
```
int((a + b*sin(e + f*x))^(1/2)/(sin(e + f*x)*(c + c*sin(e + f*x))), x)
```

3.30 $\int \frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}\,dx$

3.30.1 Optimal result

Integrand size = 33, antiderivative size = 246

$$\begin{aligned}
&\int \frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}\,dx \\
&= \frac{E\left(\frac{1}{2}\left(e-\frac{\pi}{2}+fx\right)\middle|\frac{2b}{a+b}\right)\sqrt{a+b\sin(e+fx)}}{(a-b)cf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}} \\
&\quad-\frac{\text{EllipticF}\left(\frac{1}{2}\left(e-\frac{\pi}{2}+fx\right),\frac{2b}{a+b}\right)\sqrt{\frac{a+b\sin(e+fx)}{a+b}}}{cf\sqrt{a+b\sin(e+fx)}} \\
&\quad+\frac{2\,\text{EllipticPi}\left(2,\frac{1}{2}\left(e-\frac{\pi}{2}+fx\right),\frac{2b}{a+b}\right)\sqrt{\frac{a+b\sin(e+fx)}{a+b}}}{cf\sqrt{a+b\sin(e+fx)}}+\frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{(a-b)f(c+c\sin(e+fx))}
\end{aligned}$$

output
```
cos(f*x+e)*(a+b*sin(f*x+e))^(1/2)/(a-b)/f/(c+c*sin(f*x+e))-(sin(1/2*e+1/4*
Pi+1/2*f*x)^2)^(1/2)/sin(1/2*e+1/4*Pi+1/2*f*x)*EllipticE(cos(1/2*e+1/4*Pi+
1/2*f*x),2^(1/2)*(b/(a+b))^(1/2))*(a+b*sin(f*x+e))^(1/2)/(a-b)/c/f/((a+b*s
in(f*x+e))/(a+b))^(1/2)+(sin(1/2*e+1/4*Pi+1/2*f*x)^2)^(1/2)/sin(1/2*e+1/4*
Pi+1/2*f*x)*EllipticF(cos(1/2*e+1/4*Pi+1/2*f*x),2^(1/2)*(b/(a+b))^(1/2))*(
(a+b*sin(f*x+e))/(a+b))^(1/2)/c/f/(a+b*sin(f*x+e))^(1/2)-2*(sin(1/2*e+1/4*
Pi+1/2*f*x)^2)^(1/2)/sin(1/2*e+1/4*Pi+1/2*f*x)*EllipticPi(cos(1/2*e+1/4*Pi
+1/2*f*x),2,2^(1/2)*(b/(a+b))^(1/2))*((a+b*sin(f*x+e))/(a+b))^(1/2)/c/f/(a
+b*sin(f*x+e))^(1/2)
```

3.30.2 Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 5.99 (sec) , antiderivative size = 462, normalized size of antiderivative = 1.88

$$\int \frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}\,dx$$
$$= \frac{\left(\cos\left(\frac{1}{2}(e+fx)\right)+\sin\left(\frac{1}{2}(e+fx)\right)\right)\left(-8\sin\left(\frac{1}{2}(e+fx)\right)\sqrt{a+b\sin(e+fx)}-\left(\cos\left(\frac{1}{2}(e+fx)\right)+\sin\left(\frac{1}{2}\right.\right.\right.}{}$$

input
```
Integrate[Csc[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*(c + c*Sin[e + f*x])),x]
```

output
```
((Cos[(e + f*x)/2] + Sin[(e + f*x)/2])*(-8*Sin[(e + f*x)/2]*Sqrt[a + b*Sin
[e + f*x]] - (Cos[(e + f*x)/2] + Sin[(e + f*x)/2])*(((2*I)*(-2*a*(a - b)*E
llipticE[I*ArcSinh[Sqrt[-(a + b)^(-1)]*Sqrt[a + b*Sin[e + f*x]]], (a + b)/
(a - b)] + b*(-2*a*EllipticF[I*ArcSinh[Sqrt[-(a + b)^(-1)]*Sqrt[a + b*Sin[
e + f*x]]], (a + b)/(a - b)] + b*EllipticPi[(a + b)/a, I*ArcSinh[Sqrt[-(a
+ b)^(-1)]*Sqrt[a + b*Sin[e + f*x]]], (a + b)/(a - b)]))*Sec[e + f*x]*Sqrt
[-((b*(-1 + Sin[e + f*x]))/(a + b))]*Sqrt[-((b*(1 + Sin[e + f*x]))/(a - b)
)])/(a*b*Sqrt[-(a + b)^(-1)]) - 4*Sqrt[a + b*Sin[e + f*x]] + (4*b*Elliptic
F[(-2*e + Pi - 2*f*x)/4, (2*b)/(a + b)]*Sqrt[(a + b*Sin[e + f*x])/(a + b)]
)/Sqrt[a + b*Sin[e + f*x]] + (2*(4*a - 3*b)*EllipticPi[2, (-2*e + Pi - 2*f
*x)/4, (2*b)/(a + b)]*Sqrt[(a + b*Sin[e + f*x])/(a + b)])/Sqrt[a + b*Sin[e
 + f*x]])))/(4*(a - b)*c*f*(1 + Sin[e + f*x]))
```

3.30.3 Rubi [A] (verified)

Time = 1.71 (sec) , antiderivative size = 263, normalized size of antiderivative = 1.07, number of steps used = 17, number of rules used = 17, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.515, Rules used = {3042, 3420, 3042, 3247, 27, 3042, 3231, 3042, 3134, 3042, 3132, 3142, 3042, 3140, 3286, 3042, 3284}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(e+fx)}{(c\sin(e+fx)+c)\sqrt{a+b\sin(e+fx)}}dx$$
$$\Big\downarrow \text{3042}$$

$$\int \frac{1}{\sin(e+fx)(c\sin(e+fx)+c)\sqrt{a+b\sin(e+fx)}}dx$$

$$\Big\downarrow \text{3420}$$

$$\frac{\int \frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}}dx}{c} - \int \frac{1}{\sqrt{a+b\sin(e+fx)}(\sin(e+fx)c+c)}dx$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}}dx}{c} - \int \frac{1}{\sqrt{a+b\sin(e+fx)}(\sin(e+fx)c+c)}dx$$

$$\Big\downarrow \text{3247}$$

$$-\frac{b\int -\frac{\sin(e+fx)c+c}{2\sqrt{a+b\sin(e+fx)}}dx}{c^2(a-b)} + \frac{\int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}}dx}{c} + \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}$$

$$\Big\downarrow \text{27}$$

$$\frac{b\int \frac{\sin(e+fx)c+c}{\sqrt{a+b\sin(e+fx)}}dx}{2c^2(a-b)} + \frac{\int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}}dx}{c} + \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}$$

$$\Big\downarrow \text{3042}$$

$$\frac{b\int \frac{\sin(e+fx)c+c}{\sqrt{a+b\sin(e+fx)}}dx}{2c^2(a-b)} + \frac{\int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}}dx}{c} + \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}$$

$$\Big\downarrow \text{3231}$$

$$\frac{b\left(\frac{c\int \sqrt{a+b\sin(e+fx)}dx}{b} - \frac{c(a-b)\int \frac{1}{\sqrt{a+b\sin(e+fx)}}dx}{b}\right)}{2c^2(a-b)} + \frac{\int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}}dx}{c} + \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}$$

$$\Big\downarrow \text{3042}$$

$$\frac{b\left(\frac{c\int \sqrt{a+b\sin(e+fx)}dx}{b} - \frac{c(a-b)\int \frac{1}{\sqrt{a+b\sin(e+fx)}}dx}{b}\right)}{2c^2(a-b)} + \frac{\int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}}dx}{c} + \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}$$

$$\Big\downarrow \text{3134}$$

3.30. $\int \frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}dx$

$$\frac{b\left(\frac{c\sqrt{a+b\sin(e+fx)}\int\sqrt{\frac{a}{a+b}+\frac{b\sin(e+fx)}{a+b}}dx}{b\sqrt{\frac{a+b\sin(e+fx)}{a+b}}}-\frac{c(a-b)\int\frac{1}{\sqrt{a+b\sin(e+fx)}}dx}{b}\right)}{2c^2(a-b)}+\frac{\int\frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}}dx}{c}+\frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}$$

$\Big\downarrow$ 3042

$$\frac{b\left(\frac{c\sqrt{a+b\sin(e+fx)}\int\sqrt{\frac{a}{a+b}+\frac{b\sin(e+fx)}{a+b}}dx}{b\sqrt{\frac{a+b\sin(e+fx)}{a+b}}}-\frac{c(a-b)\int\frac{1}{\sqrt{a+b\sin(e+fx)}}dx}{b}\right)}{2c^2(a-b)}+\frac{\int\frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}}dx}{c}+\frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}$$

$\Big\downarrow$ 3132

$$\frac{b\left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right)|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}}-\frac{c(a-b)\int\frac{1}{\sqrt{a+b\sin(e+fx)}}dx}{b}\right)}{2c^2(a-b)}+\frac{\int\frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}}dx}{c}+\frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}$$

$\Big\downarrow$ 3142

$$\frac{b\left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right)|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}}-\frac{c(a-b)\sqrt{\frac{a+b\sin(e+fx)}{a+b}}\int\frac{1}{\sqrt{\frac{a}{a+b}+\frac{b\sin(e+fx)}{a+b}}}dx}{b\sqrt{a+b\sin(e+fx)}}\right)}{2c^2(a-b)}+\frac{\int\frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}}dx}{c}+\frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}$$

$\Big\downarrow$ 3042

$$\frac{b\left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right)|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}}-\frac{c(a-b)\sqrt{\frac{a+b\sin(e+fx)}{a+b}}\int\frac{1}{\sqrt{\frac{a}{a+b}+\frac{b\sin(e+fx)}{a+b}}}dx}{b\sqrt{a+b\sin(e+fx)}}\right)}{2c^2(a-b)}+\frac{\int\frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}}dx}{c}+\frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}$$

$\Big\downarrow$ 3140

3.30. $\int\frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}dx$

$$\frac{\int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}}dx}{c}+\frac{b\left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right)|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}}-\frac{2c(a-b)\sqrt{\frac{a+b\sin(e+fx)}{a+b}}\operatorname{EllipticF}\left(\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right),\frac{2b}{a+b}\right)}{bf\sqrt{a+b\sin(e+fx)}}\right)}{2c^2(a-b)}+\frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}$$

$$\Big\downarrow \; 3286$$

$$\frac{\sqrt{\frac{a+b\sin(e+fx)}{a+b}}\int \frac{\csc(e+fx)}{\sqrt{\frac{a}{a+b}+\frac{b\sin(e+fx)}{a+b}}}dx}{c\sqrt{a+b\sin(e+fx)}}+\frac{b\left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right)|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}}-\frac{2c(a-b)\sqrt{\frac{a+b\sin(e+fx)}{a+b}}\operatorname{EllipticF}\left(\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right),\frac{2b}{a+b}\right)}{bf\sqrt{a+b\sin(e+fx)}}\right)}{2c^2(a-b)}+\frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}$$

$$\Big\downarrow \; 3042$$

$$\frac{\sqrt{\frac{a+b\sin(e+fx)}{a+b}}\int \frac{1}{\sin(e+fx)\sqrt{\frac{a}{a+b}+\frac{b\sin(e+fx)}{a+b}}}dx}{c\sqrt{a+b\sin(e+fx)}}+\frac{b\left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right)|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}}-\frac{2c(a-b)\sqrt{\frac{a+b\sin(e+fx)}{a+b}}\operatorname{EllipticF}\left(\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right),\frac{2b}{a+b}\right)}{bf\sqrt{a+b\sin(e+fx)}}\right)}{2c^2(a-b)}+\frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}$$

$$\Big\downarrow \; 3284$$

$$\frac{b\left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right)|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}}-\frac{2c(a-b)\sqrt{\frac{a+b\sin(e+fx)}{a+b}}\operatorname{EllipticF}\left(\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right),\frac{2b}{a+b}\right)}{bf\sqrt{a+b\sin(e+fx)}}\right)}{2c^2(a-b)}+\frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}+\frac{2\sqrt{\frac{a+b\sin(e+fx)}{a+b}}\operatorname{EllipticPi}\left(2,\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right),\frac{2b}{a+b}\right)}{cf\sqrt{a+b\sin(e+fx)}}$$

input
```
Int[Csc[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*(c + c*Sin[e + f*x])),x]
```

3.30. $\int \frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}dx$

output
```
(2*EllipticPi[2, (e - Pi/2 + f*x)/2, (2*b)/(a + b)]*Sqrt[(a + b*Sin[e + f*
x])/(a + b)])/(c*f*Sqrt[a + b*Sin[e + f*x]]) + (Cos[e + f*x]*Sqrt[a + b*Si
n[e + f*x]])/((a - b)*f*(c + c*Sin[e + f*x])) + (b*((2*c*EllipticE[(e - Pi
/2 + f*x)/2, (2*b)/(a + b)]*Sqrt[a + b*Sin[e + f*x]])/(b*f*Sqrt[(a + b*Sin
[e + f*x])/(a + b)]) - (2*(a - b)*c*EllipticF[(e - Pi/2 + f*x)/2, (2*b)/(a
 + b)]*Sqrt[(a + b*Sin[e + f*x])/(a + b)])/(b*f*Sqrt[a + b*Sin[e + f*x]]))
)/(2*(a - b)*c^2)
```

3.30.3.1 Defintions of rubi rules used

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3132
```
Int[Sqrt[(a_) + (b_.)*sin[(c_.) + (d_.)*(x_)]], x_Symbol] :> Simp[2*(Sqrt[a
 + b]/d)*EllipticE[(1/2)*(c - Pi/2 + d*x), 2*(b/(a + b))], x] /; FreeQ[{a,
b, c, d}, x] && NeQ[a^2 - b^2, 0] && GtQ[a + b, 0]
```

rule 3134
```
Int[Sqrt[(a_) + (b_.)*sin[(c_.) + (d_.)*(x_)]], x_Symbol] :> Simp[Sqrt[a +
b*Sin[c + d*x]]/Sqrt[(a + b*Sin[c + d*x])/(a + b)]   Int[Sqrt[a/(a + b) + (
b/(a + b))*Sin[c + d*x]], x], x] /; FreeQ[{a, b, c, d}, x] && NeQ[a^2 - b^2
, 0] &&  !GtQ[a + b, 0]
```

rule 3140
```
Int[1/Sqrt[(a_) + (b_.)*sin[(c_.) + (d_.)*(x_)]], x_Symbol] :> Simp[(2/(d*S
qrt[a + b]))*EllipticF[(1/2)*(c - Pi/2 + d*x), 2*(b/(a + b))], x] /; FreeQ[
{a, b, c, d}, x] && NeQ[a^2 - b^2, 0] && GtQ[a + b, 0]
```

rule 3142
```
Int[1/Sqrt[(a_) + (b_.)*sin[(c_.) + (d_.)*(x_)]], x_Symbol] :> Simp[Sqrt[(a
 + b*Sin[c + d*x])/(a + b)]/Sqrt[a + b*Sin[c + d*x]]   Int[1/Sqrt[a/(a + b)
 + (b/(a + b))*Sin[c + d*x]], x], x] /; FreeQ[{a, b, c, d}, x] && NeQ[a^2 -
 b^2, 0] &&  !GtQ[a + b, 0]
```

rule 3231
```
Int[((c_.) + (d_.)*sin[(e_.) + (f_.)*(x_)])/Sqrt[(a_) + (b_.)*sin[(e_.) + (
f_.)*(x_)]], x_Symbol] :> Simp[(b*c - a*d)/b   Int[1/Sqrt[a + b*Sin[e + f*x
]], x], x] + Simp[d/b   Int[Sqrt[a + b*Sin[e + f*x]], x], x] /; FreeQ[{a, b
, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0]
```

rule 3247
```
Int[((c_.) + (d_.)*sin[(e_.) + (f_.)*(x_)])^(n_)/((a_) + (b_.)*sin[(e_.) +
(f_.)*(x_)]), x_Symbol] :> Simp[(-b^2)*Cos[e + f*x]*((c + d*Sin[e + f*x])^(
n + 1)/(a*f*(b*c - a*d)*(a + b*Sin[e + f*x]))), x] + Simp[d/(a*(b*c - a*d))
   Int[(c + d*Sin[e + f*x])^n*(a*n - b*(n + 1)*Sin[e + f*x]), x], x] /; Fre
eQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0] && NeQ[
c^2 - d^2, 0] && LtQ[n, 0] && (IntegerQ[2*n] || EqQ[c, 0])
```

rule 3284
```
Int[1/(((a_.) + (b_.)*sin[(e_.) + (f_.)*(x_)])*Sqrt[(c_.) + (d_.)*sin[(e_.)
 + (f_.)*(x_)]]), x_Symbol] :> Simp[(2/(f*(a + b)*Sqrt[c + d]))*EllipticPi[
2*(b/(a + b)), (1/2)*(e - Pi/2 + f*x), 2*(d/(c + d))], x] /; FreeQ[{a, b, c
, d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2,
0] && GtQ[c + d, 0]
```

rule 3286
```
Int[1/(((a_.) + (b_.)*sin[(e_.) + (f_.)*(x_)])*Sqrt[(c_.) + (d_.)*sin[(e_.)
 + (f_.)*(x_)]]), x_Symbol] :> Simp[Sqrt[(c + d*Sin[e + f*x])/(c + d)]/Sqrt
[c + d*Sin[e + f*x]]   Int[1/((a + b*Sin[e + f*x])*Sqrt[c/(c + d) + (d/(c +
 d))*Sin[e + f*x]]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*
d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0] &&  !GtQ[c + d, 0]
```

rule 3420
```
Int[1/(sin[(e_.) + (f_.)*(x_)]*Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]*(
(c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[1/c   Int[1/(Sin[
e + f*x]*Sqrt[a + b*Sin[e + f*x]]), x], x] - Simp[d/c   Int[1/(Sqrt[a + b*S
in[e + f*x]]*(c + d*Sin[e + f*x])), x], x] /; FreeQ[{a, b, c, d, e, f}, x]
&& NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0]
```

3.30. $\int \frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}\,dx$

3.30.4 Maple [A] (verified)

Time = 1.25 (sec) , antiderivative size = 587, normalized size of antiderivative = 2.39

method	result
default	$\frac{\sqrt{-(-b\sin(fx+e)-a)(\cos^2(fx+e))}\left(-\frac{2\left(\frac{a}{b}-1\right)\sqrt{\frac{a+b\sin(fx+e)}{a-b}}\sqrt{\frac{(1-\sin(fx+e))b}{a+b}}\sqrt{\frac{(-\sin(fx+e)-1)b}{a-b}}\,b\Pi\left(\sqrt{\frac{a+b\sin(fx+e)}{a-b}},-\frac{\left(-\frac{a}{b}+1\right)b}{a}\right.}{\sqrt{-(-b\sin(fx+e)-a)\left(\cos^2(fx+e)\right)}\,a}\right.}{}$

input
```
int(1/sin(f*x+e)/(c+c*sin(f*x+e))/(a+b*sin(f*x+e))^(1/2),x,method=_RETURNV
ERBOSE)
```

output
```
(-(-b*sin(f*x+e)-a)*cos(f*x+e)^2)^(1/2)/c*(-2*(1/b*a-1)*((a+b*sin(f*x+e))/
(a-b))^(1/2)*(1/(a+b)*(1-sin(f*x+e))*b)^(1/2)*(1/(a-b)*(-sin(f*x+e)-1)*b)^
(1/2)/(-(-b*sin(f*x+e)-a)*cos(f*x+e)^2)^(1/2)*b/a*EllipticPi(((a+b*sin(f*x
+e))/(a-b))^(1/2),-(-1/b*a+1)*b/a,((a-b)/(a+b))^(1/2))+(-b*sin(f*x+e)^2-a*
sin(f*x+e)+b*sin(f*x+e)+a)/(a-b)/((1+sin(f*x+e))*(sin(f*x+e)-1)*(-b*sin(f*
x+e)-a))^(1/2)+2*b/(2*a-2*b)*(1/b*a-1)*((a+b*sin(f*x+e))/(a-b))^(1/2)*(1/(
a+b)*(1-sin(f*x+e))*b)^(1/2)*(1/(a-b)*(-sin(f*x+e)-1)*b)^(1/2)/(-(-b*sin(f
*x+e)-a)*cos(f*x+e)^2)^(1/2)*EllipticF(((a+b*sin(f*x+e))/(a-b))^(1/2),((a-
b)/(a+b))^(1/2))+b/(a-b)*(1/b*a-1)*((a+b*sin(f*x+e))/(a-b))^(1/2)*(1/(a+b)
*(1-sin(f*x+e))*b)^(1/2)*(1/(a-b)*(-sin(f*x+e)-1)*b)^(1/2)/(-(-b*sin(f*x+e
)-a)*cos(f*x+e)^2)^(1/2)*((-1/b*a-1)*EllipticE(((a+b*sin(f*x+e))/(a-b))^(1
/2),((a-b)/(a+b))^(1/2))+EllipticF(((a+b*sin(f*x+e))/(a-b))^(1/2),((a-b)/(
a+b))^(1/2))))/cos(f*x+e)/(a+b*sin(f*x+e))^(1/2)/f
```

3.30.5 Fricas [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}\,dx = \text{Timed out}$$

input
```
integrate(1/sin(f*x+e)/(c+c*sin(f*x+e))/(a+b*sin(f*x+e))^(1/2),x, algorith
m="fricas")
```

output
```
Timed out
```

3.30.6 Sympy [F]

$$\int \frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}\,dx = \frac{\int \frac{1}{\sqrt{a+b\sin{(e+fx)}}\sin^2{(e+fx)}+\sqrt{a+b\sin{(e+fx)}}\sin{(e+fx)}}\,dx}{c}$$

input
```
integrate(1/sin(f*x+e)/(c+c*sin(f*x+e))/(a+b*sin(f*x+e))**(1/2),x)
```

output
```
Integral(1/(sqrt(a + b*sin(e + f*x))*sin(e + f*x)**2 + sqrt(a + b*sin(e +
f*x))*sin(e + f*x)), x)/c
```

3.30.7 Maxima [F]

$$\int \frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}\,dx = \int \frac{1}{\sqrt{b\sin{(fx+e)}+a}(c\sin{(fx+e)}+c)\sin{(fx+e)}}\,dx$$

input
```
integrate(1/sin(f*x+e)/(c+c*sin(f*x+e))/(a+b*sin(f*x+e))^(1/2),x, algorith
m="maxima")
```

output
```
integrate(1/(sqrt(b*sin(f*x + e) + a)*(c*sin(f*x + e) + c)*sin(f*x + e)),
x)
```

3.30.8 Giac [F]

$$\int \frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}\,dx = \int \frac{1}{\sqrt{b\sin{(fx+e)}+a}(c\sin{(fx+e)}+c)\sin{(fx+e)}}\,dx$$

input
```
integrate(1/sin(f*x+e)/(c+c*sin(f*x+e))/(a+b*sin(f*x+e))^(1/2),x, algorith
m="giac")
```

output
```
integrate(1/(sqrt(b*sin(f*x + e) + a)*(c*sin(f*x + e) + c)*sin(f*x + e)),
x)
```

3.30.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}\,dx$$
$$=\int \frac{1}{\sin\left(e+f\,x\right)\,\sqrt{a+b\,\sin\left(e+f\,x\right)}\,\left(c+c\,\sin\left(e+f\,x\right)\right)}\,dx$$

input
```
int(1/(sin(e + f*x)*(a + b*sin(e + f*x))^(1/2)*(c + c*sin(e + f*x))),x)
```

output
```
int(1/(sin(e + f*x)*(a + b*sin(e + f*x))^(1/2)*(c + c*sin(e + f*x))), x)
```

3.31 $\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}{c+c\sin(e+fx)}\,dx$

3.31.1 Optimal result

Integrand size = 39, antiderivative size = 267

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}{c+c\sin(e+fx)}\,dx$$

$$= \frac{2\sqrt{g}\,\text{EllipticPi}\left(\frac{b}{a+b},\arcsin\left(\frac{\sqrt{a+b}\sqrt{g\sin(e+fx)}}{\sqrt{g}\sqrt{a+b\sin(e+fx)}}\right),-\frac{a-b}{a+b}\right)\sec(e+fx)\sqrt{\frac{a(1-\sin(e+fx))}{a+b\sin(e+fx)}}\sqrt{\frac{a(1+\sin(e+fx))}{a+b\sin(e+fx)}}(a+b\sin}{\sqrt{a+b}cf}$$

$$+\frac{gE\left(\arcsin\left(\frac{\cos(e+fx)}{1+\sin(e+fx)}\right)|-\frac{a-b}{a+b}\right)\sqrt{\frac{\sin(e+fx)}{1+\sin(e+fx)}}\sqrt{a+b\sin(e+fx)}}{cf\sqrt{g\sin(e+fx)}\sqrt{\frac{a+b\sin(e+fx)}{(a+b)(1+\sin(e+fx))}}}$$

output
```
2*EllipticPi((a+b)^(1/2)*(g*sin(f*x+e))^(1/2)/g^(1/2)/(a+b*sin(f*x+e))^(1/
2),b/(a+b),((-a+b)/(a+b))^(1/2))*sec(f*x+e)*(a+b*sin(f*x+e))*g^(1/2)*(a*(1
-sin(f*x+e))/(a+b*sin(f*x+e)))^(1/2)*(a*(1+sin(f*x+e))/(a+b*sin(f*x+e)))^(
1/2)/c/f/(a+b)^(1/2)+g*EllipticE(cos(f*x+e)/(1+sin(f*x+e)),((-a+b)/(a+b))^
(1/2))*(sin(f*x+e)/(1+sin(f*x+e)))^(1/2)*(a+b*sin(f*x+e))^(1/2)/c/f/(g*sin
(f*x+e))^(1/2)/((a+b*sin(f*x+e))/(a+b)/(1+sin(f*x+e)))^(1/2)
```

3.31.2 Mathematica [C] (warning: unable to verify)

Result contains complex when optimal does not.

Time = 32.78 (sec) , antiderivative size = 13199, normalized size of antiderivative = 49.43

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}{c+c\sin(e+fx)}\,dx = \text{Result too large to show}$$

input
```
Integrate[(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]])/(c + c*Sin[e + f
*x]),x]
```

output
```
Result too large to show
```

3.31.3 Rubi [A] (verified)

Time = 0.96 (sec) , antiderivative size = 267, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.128, Rules used = {3042, 3407, 3042, 3290, 3411}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}{c\sin(e+fx)+c}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}{c\sin(e+fx)+c}dx$$

$$\Big\downarrow \text{3407}$$

$$\frac{g\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}}dx}{c}-g\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(\sin(e+fx)c+c)}dx$$

$$\Big\downarrow \text{3042}$$

$$\frac{g\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}}dx}{c}-g\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(\sin(e+fx)c+c)}dx$$

$$\Big\downarrow \text{3290}$$

$$\frac{2\sqrt{g}\sec(e+fx)\sqrt{\frac{a(1-\sin(e+fx))}{a+b\sin(e+fx)}}\sqrt{\frac{a(\sin(e+fx)+1)}{a+b\sin(e+fx)}}(a+b\sin(e+fx))\operatorname{EllipticPi}\left(\frac{b}{a+b},\arcsin\left(\frac{\sqrt{a+b}\sqrt{g\sin(e+fx)}}{\sqrt{g}\sqrt{a+b\sin(e+fx)}}\right),-\frac{a-}{a+}\right.}{cf\sqrt{a+b}}$$

$$g\int\frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(\sin(e+fx)c+c)}dx$$

$$\Big\downarrow \text{3411}$$

$$\frac{g\sqrt{\frac{\sin(e+fx)}{\sin(e+fx)+1}}\sqrt{a+b\sin(e+fx)}E\left(\arcsin\left(\frac{\cos(e+fx)}{\sin(e+fx)+1}\right)|-\frac{a-b}{a+b}\right)}{cf\sqrt{g\sin(e+fx)}\sqrt{\frac{a+b\sin(e+fx)}{(a+b)(\sin(e+fx)+1)}}}+$$

$$\frac{2\sqrt{g}\sec(e+fx)\sqrt{\frac{a(1-\sin(e+fx))}{a+b\sin(e+fx)}}\sqrt{\frac{a(\sin(e+fx)+1)}{a+b\sin(e+fx)}}(a+b\sin(e+fx))\operatorname{EllipticPi}\left(\frac{b}{a+b},\arcsin\left(\frac{\sqrt{a+b}\sqrt{g\sin(e+fx)}}{\sqrt{g}\sqrt{a+b\sin(e+fx)}}\right),-\frac{a-}{a+}\right.}{cf\sqrt{a+b}}$$

input
```
Int[(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]])/(c + c*Sin[e + f*x]),x
]
```

output
```
(2*Sqrt[g]*EllipticPi[b/(a + b), ArcSin[(Sqrt[a + b]*Sqrt[g*Sin[e + f*x]])
/(Sqrt[g]*Sqrt[a + b*Sin[e + f*x]])], -((a - b)/(a + b))]*Sec[e + f*x]*Sqr
t[(a*(1 - Sin[e + f*x]))/(a + b*Sin[e + f*x])]*Sqrt[(a*(1 + Sin[e + f*x]))
/(a + b*Sin[e + f*x])]*(a + b*Sin[e + f*x]))/(Sqrt[a + b]*c*f) + (g*Ellipt
icE[ArcSin[Cos[e + f*x]/(1 + Sin[e + f*x])], -((a - b)/(a + b))]*Sqrt[Sin[
e + f*x]/(1 + Sin[e + f*x])]*Sqrt[a + b*Sin[e + f*x]])/(c*f*Sqrt[g*Sin[e +
 f*x]]*Sqrt[(a + b*Sin[e + f*x])/((a + b)*(1 + Sin[e + f*x]))])
```

3.31.3.1 Defintions of rubi rules used

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3290
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/Sqrt[(c_.) + (d_.)*sin[(e_.)
 + (f_.)*(x_)]], x_Symbol] :> Simp[2*((a + b*Sin[e + f*x])/(d*f*Rt[(a + b)/
(c + d), 2]*Cos[e + f*x]))*Sqrt[(b*c - a*d)*((1 + Sin[e + f*x])/((c - d)*(a
 + b*Sin[e + f*x])))]*Sqrt[(-(b*c - a*d))*((1 - Sin[e + f*x])/((c + d)*(a +
 b*Sin[e + f*x])))]*EllipticPi[b*((c + d)/(d*(a + b))), ArcSin[Rt[(a + b)/(
c + d), 2]*(Sqrt[c + d*Sin[e + f*x]]/Sqrt[a + b*Sin[e + f*x]])], (a - b)*((
c + d)/((a + b)*(c - d)))], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c -
 a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0] && PosQ[(a + b)/(c + d)]
```

rule 3407
```
Int[(Sqrt[(g_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.
)*(x_)]])/((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] :> Simp[g/d   I
nt[Sqrt[a + b*Sin[e + f*x]]/Sqrt[g*Sin[e + f*x]], x], x] - Simp[c*(g/d)   I
nt[Sqrt[a + b*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c + d*Sin[e + f*x])), x]
, x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && (EqQ[a^2 -
b^2, 0] || EqQ[c^2 - d^2, 0])
```

rule 3411
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(g_.)*sin[(e_.) + (f_.
)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[(-Sqrt[
a + b*Sin[e + f*x]])*(Sqrt[d*(Sin[e + f*x]/(c + d*Sin[e + f*x]))]/(d*f*Sqrt
[g*Sin[e + f*x]]*Sqrt[c^2*((a + b*Sin[e + f*x])/((a*c + b*d)*(c + d*Sin[e +
 f*x])))]))*EllipticE[ArcSin[c*(Cos[e + f*x]/(c + d*Sin[e + f*x]))], (b*c -
 a*d)/(b*c + a*d)], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d,
 0] && NeQ[a^2 - b^2, 0] && EqQ[c^2 - d^2, 0]
```

3.31.4 Maple [C] (warning: unable to verify)

Result contains complex when optimal does not.

Time = 2.60 (sec) , antiderivative size = 9956, normalized size of antiderivative = 37.29

method	result	size
default	Expression too large to display	9956

input
```
int((g*sin(f*x+e))^(1/2)*(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x,method=
_RETURNVERBOSE)
```

output
```
result too large to display
```

3.31.5 Fricas [F]

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}{c+c\sin(e+fx)}\,dx = \int \frac{\sqrt{b\sin\left(fx+e\right)+a}\sqrt{g\sin\left(fx+e\right)}}{c\sin\left(fx+e\right)+c}\,dx$$

input
```
integrate((g*sin(f*x+e))^(1/2)*(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x,
algorithm="fricas")
```

output
```
integral(sqrt(b*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))/(c*sin(f*x + e) + c
), x)
```

3.31.6 Sympy [F]

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}{c+c\sin(e+fx)}\,dx = \frac{\int \frac{\sqrt{g\sin{(e+fx)}}\sqrt{a+b\sin{(e+fx)}}}{\sin{(e+fx)}+1}\,dx}{c}$$

input
```
integrate((g*sin(f*x+e))**(1/2)*(a+b*sin(f*x+e))**(1/2)/(c+c*sin(f*x+e)),x
)
```

output
```
Integral(sqrt(g*sin(e + f*x))*sqrt(a + b*sin(e + f*x))/(sin(e + f*x) + 1),
 x)/c
```

3.31.7 Maxima [F]

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}{c+c\sin(e+fx)}\,dx = \int \frac{\sqrt{b\sin{(fx+e)}+a}\sqrt{g\sin{(fx+e)}}}{c\sin{(fx+e)}+c}\,dx$$

input
```
integrate((g*sin(f*x+e))^(1/2)*(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x,
algorithm="maxima")
```

output
```
integrate(sqrt(b*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))/(c*sin(f*x + e) +
c), x)
```

3.31.8 Giac [F]

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}{c+c\sin(e+fx)}\,dx = \int \frac{\sqrt{b\sin{(fx+e)}+a}\sqrt{g\sin{(fx+e)}}}{c\sin{(fx+e)}+c}\,dx$$

input
```
integrate((g*sin(f*x+e))^(1/2)*(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x,
algorithm="giac")
```

output
```
integrate(sqrt(b*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))/(c*sin(f*x + e) +
c), x)
```

3.31.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}{c+c\sin(e+fx)}\,dx=\int \frac{\sqrt{g\,\sin\left(e+f\,x\right)}\,\sqrt{a+b\,\sin\left(e+f\,x\right)}}{c+c\,\sin\left(e+f\,x\right)}\,dx$$

input
```
int(((g*sin(e + f*x))^(1/2)*(a + b*sin(e + f*x))^(1/2))/(c + c*sin(e + f*x
)),x)
```

output
```
int(((g*sin(e + f*x))^(1/2)*(a + b*sin(e + f*x))^(1/2))/(c + c*sin(e + f*x
)), x)
```

3.32 $\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c+c\sin(e+fx))}\,dx$

3.32.1 Optimal result

Integrand size = 39, antiderivative size = 116

$$\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c+c\sin(e+fx))}\,dx = -\frac{E\left(\arcsin\left(\frac{\cos(e+fx)}{1+\sin(e+fx)}\right)\middle|-\frac{a-b}{a+b}\right)\sqrt{\frac{\sin(e+fx)}{1+\sin(e+fx)}}\sqrt{a+b\sin(e+fx)}}{cf\sqrt{g\sin(e+fx)}\sqrt{\frac{a+b\sin(e+fx)}{(a+b)(1+\sin(e+fx))}}}$$

output
```
-EllipticE(cos(f*x+e)/(1+sin(f*x+e)),((-a+b)/(a+b))^(1/2))*(sin(f*x+e)/(1+
sin(f*x+e)))^(1/2)*(a+b*sin(f*x+e))^(1/2)/c/f/(g*sin(f*x+e))^(1/2)/((a+b*s
in(f*x+e))/(a+b)/(1+sin(f*x+e)))^(1/2)
```

3.32.2 Mathematica [B] (warning: unable to verify)

Leaf count is larger than twice the leaf count of optimal. 3415 vs. 2(116) = 232.

Time = 28.98 (sec) , antiderivative size = 3415, normalized size of antiderivative = 29.44

$$\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c+c\sin(e+fx))}\,dx = \text{Result too large to show}$$

input
```
Integrate[Sqrt[a + b*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c + c*Sin[e + f*
x])),x]
```

output
```
(-2*Sin[(e + f*x)/2]*(Cos[(e + f*x)/2] + Sin[(e + f*x)/2])*Sin[e + f*x]*Sq
rt[a + b*Sin[e + f*x]])/(f*Sqrt[g*Sin[e + f*x]]*(c + c*Sin[e + f*x])) + ((
Cos[(e + f*x)/2] + Sin[(e + f*x)/2])^2*Sqrt[a + b*Sin[e + f*x]]*((a*Sqrt[S
in[e + f*x]])/(2*Sqrt[a + b*Sin[e + f*x]]) - (b*Sqrt[Sin[e + f*x]])/(2*Sqr
t[a + b*Sin[e + f*x]]) + (a*Cot[(e + f*x)/2]*Sqrt[Sin[e + f*x]])/(2*Sqrt[a
 + b*Sin[e + f*x]]) + (b*Cot[(e + f*x)/2]*Sqrt[Sin[e + f*x]])/(2*Sqrt[a +
b*Sin[e + f*x]]) - (b*Cos[(3*(e + f*x))/2]*Csc[(e + f*x)/2]*Sqrt[Sin[e + f
*x]])/(2*Sqrt[a + b*Sin[e + f*x]]) + (b*Csc[(e + f*x)/2]*Sqrt[Sin[e + f*x]
]*Sin[(3*(e + f*x))/2])/(2*Sqrt[a + b*Sin[e + f*x]]))*(1 - Cos[e + f*x] +
Sin[e + f*x] - (2*a*(EllipticE[ArcSin[Sqrt[(-b + Sqrt[-a^2 + b^2] - a*Tan[
(e + f*x)/2])/Sqrt[-a^2 + b^2]]/Sqrt[2]], (2*Sqrt[-a^2 + b^2])/(-b + Sqrt[
-a^2 + b^2])]*Tan[(e + f*x)/2] + EllipticF[ArcSin[Sqrt[(b + Sqrt[-a^2 + b^
2] + a*Tan[(e + f*x)/2])/Sqrt[-a^2 + b^2]]/Sqrt[2]], (2*Sqrt[-a^2 + b^2])/
(b + Sqrt[-a^2 + b^2])]*Sqrt[(a*Tan[(e + f*x)/2])/(-b + Sqrt[-a^2 + b^2])]
*Sqrt[-((a*Tan[(e + f*x)/2])/(b + Sqrt[-a^2 + b^2]))]))/(Sqrt[-a^2 + b^2]*
Sqrt[(a*Sec[(e + f*x)/2]^2*(a + b*Sin[e + f*x]))/(a^2 - b^2)]*Sqrt[(a*Tan[
(e + f*x)/2])/(-b + Sqrt[-a^2 + b^2])])))/(f*Sqrt[g*Sin[e + f*x]]*(c + c*S
in[e + f*x])*((b*Cos[e + f*x]*(1 - Cos[e + f*x] + Sin[e + f*x] - (2*a*(Ell
ipticE[ArcSin[Sqrt[(-b + Sqrt[-a^2 + b^2] - a*Tan[(e + f*x)/2])/Sqrt[-a^2
+ b^2]]/Sqrt[2]], (2*Sqrt[-a^2 + b^2])/(-b + Sqrt[-a^2 + b^2])]*Tan[(e ...
```

3.32.3 Rubi [A] (verified)

Time = 0.40 (sec) , antiderivative size = 116, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.051, Rules used = {3042, 3411}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sqrt{a+b\sin(e+fx)}}{(c\sin(e+fx)+c)\sqrt{g\sin(e+fx)}}dx$$

$$\Big\downarrow \text{ 3042}$$

$$\int \frac{\sqrt{a+b\sin(e+fx)}}{(c\sin(e+fx)+c)\sqrt{g\sin(e+fx)}}dx$$

$$\Big\downarrow \text{ 3411}$$

$$-\frac{\sqrt{\frac{\sin(e+fx)}{\sin(e+fx)+1}}\sqrt{a+b\sin(e+fx)}E\left(\arcsin\left(\frac{\cos(e+fx)}{\sin(e+fx)+1}\right)|-\frac{a-b}{a+b}\right)}{cf\sqrt{g\sin(e+fx)}\sqrt{\frac{a+b\sin(e+fx)}{(a+b)(\sin(e+fx)+1)}}}$$

input
```
Int[Sqrt[a + b*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c + c*Sin[e + f*x])),x
]
```

output
```
-((EllipticE[ArcSin[Cos[e + f*x]/(1 + Sin[e + f*x])], -((a - b)/(a + b))]*
Sqrt[Sin[e + f*x]/(1 + Sin[e + f*x])]*Sqrt[a + b*Sin[e + f*x]])/(c*f*Sqrt[
g*Sin[e + f*x]]*Sqrt[(a + b*Sin[e + f*x])/((a + b)*(1 + Sin[e + f*x]))]))
```

3.32.3.1 Defintions of rubi rules used

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3411
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(g_.)*sin[(e_.) + (f_.
)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[(-Sqrt[
a + b*Sin[e + f*x]])*(Sqrt[d*(Sin[e + f*x]/(c + d*Sin[e + f*x]))]/(d*f*Sqrt
[g*Sin[e + f*x]]*Sqrt[c^2*((a + b*Sin[e + f*x])/((a*c + b*d)*(c + d*Sin[e +
 f*x])))]))*EllipticE[ArcSin[c*(Cos[e + f*x]/(c + d*Sin[e + f*x]))], (b*c -
 a*d)/(b*c + a*d)], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d,
 0] && NeQ[a^2 - b^2, 0] && EqQ[c^2 - d^2, 0]
```

3.32.4 Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 3006 vs. $2(108) = 216$.

Time = 2.16 (sec) , antiderivative size = 3007, normalized size of antiderivative = 25.92

method	result	size
default	Expression too large to display	3007

input
```
int((a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e))/(g*sin(f*x+e))^(1/2),x,method=
_RETURNVERBOSE)
```

output
```
1/2/c/f/(g/((1-cos(f*x+e))^2*csc(f*x+e)^2+1)*(csc(f*x+e)-cot(f*x+e)))^(1/2
)*((a*(1-cos(f*x+e))^2*csc(f*x+e)^2+2*b*(csc(f*x+e)-cot(f*x+e))+a)/((1-cos
(f*x+e))^2*csc(f*x+e)^2+1))^(1/2)*((1/(b+(-a^2+b^2)^(1/2))*(a*(csc(f*x+e)-
cot(f*x+e))+(-a^2+b^2)^(1/2)+b))^(1/2)*(1/(-a^2+b^2)^(1/2)*(-a*(csc(f*x+e)
-cot(f*x+e))+(-a^2+b^2)^(1/2)-b))^(1/2)*(-a/(b+(-a^2+b^2)^(1/2))*(csc(f*x+
e)-cot(f*x+e)))^(1/2)*EllipticF((1/(b+(-a^2+b^2)^(1/2))*(a*(csc(f*x+e)-cot
(f*x+e))+(-a^2+b^2)^(1/2)+b))^(1/2),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^
2+b^2)^(1/2))^(1/2))*(-a^2+b^2)^(1/2)*2^(1/2)*a*(csc(f*x+e)-cot(f*x+e))+(1
/(b+(-a^2+b^2)^(1/2))*(a*(csc(f*x+e)-cot(f*x+e))+(-a^2+b^2)^(1/2)+b))^(1/2
)*(1/(-a^2+b^2)^(1/2)*(-a*(csc(f*x+e)-cot(f*x+e))+(-a^2+b^2)^(1/2)-b))^(1/
2)*(-a/(b+(-a^2+b^2)^(1/2))*(csc(f*x+e)-cot(f*x+e)))^(1/2)*EllipticF((1/(b
+(-a^2+b^2)^(1/2))*(a*(csc(f*x+e)-cot(f*x+e))+(-a^2+b^2)^(1/2)+b))^(1/2),1
/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1/2))*2^(1/2)*a^2*(csc
(f*x+e)-cot(f*x+e))+(1/(b+(-a^2+b^2)^(1/2))*(a*(csc(f*x+e)-cot(f*x+e))+(-a
^2+b^2)^(1/2)+b))^(1/2)*(1/(-a^2+b^2)^(1/2)*(-a*(csc(f*x+e)-cot(f*x+e))+(-
a^2+b^2)^(1/2)-b))^(1/2)*(-a/(b+(-a^2+b^2)^(1/2))*(csc(f*x+e)-cot(f*x+e)))
^(1/2)*EllipticF((1/(b+(-a^2+b^2)^(1/2))*(a*(csc(f*x+e)-cot(f*x+e))+(-a^2+
b^2)^(1/2)+b))^(1/2),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(
1/2))*2^(1/2)*a*b*(csc(f*x+e)-cot(f*x+e))+2*(1/(b+(-a^2+b^2)^(1/2))*(a*(cs
c(f*x+e)-cot(f*x+e))+(-a^2+b^2)^(1/2)+b))^(1/2)*(1/(-a^2+b^2)^(1/2)*(-a...
```

3.32.5 Fricas [F]

$$\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c+c\sin(e+fx))}\,dx = \int \frac{\sqrt{b\sin(fx+e)+a}}{(c\sin(fx+e)+c)\sqrt{g\sin(fx+e)}}\,dx$$

input
```
integrate((a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e))/(g*sin(f*x+e))^(1/2),x,
algorithm="fricas")
```

output
```
integral(-sqrt(b*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))/(c*g*cos(f*x + e)^
2 - c*g*sin(f*x + e) - c*g), x)
```

3.32.6 Sympy [F]

$$\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c+c\sin(e+fx))}\,dx = \frac{\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}\sin(e+fx)+\sqrt{g\sin(e+fx)}}\,dx}{c}$$

input
```
integrate((a+b*sin(f*x+e))**(1/2)/(c+c*sin(f*x+e))/(g*sin(f*x+e))**(1/2),x
)
```

output
```
Integral(sqrt(a + b*sin(e + f*x))/(sqrt(g*sin(e + f*x))*sin(e + f*x) + sqr
t(g*sin(e + f*x))), x)/c
```

3.32.7 Maxima [F]

$$\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c+c\sin(e+fx))}\,dx = \int \frac{\sqrt{b\sin(fx+e)+a}}{(c\sin(fx+e)+c)\sqrt{g\sin(fx+e)}}\,dx$$

input
```
integrate((a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e))/(g*sin(f*x+e))^(1/2),x,
algorithm="maxima")
```

output
```
integrate(sqrt(b*sin(f*x + e) + a)/((c*sin(f*x + e) + c)*sqrt(g*sin(f*x +
e))), x)
```

3.32.8 Giac [F]

$$\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c+c\sin(e+fx))}\,dx = \int \frac{\sqrt{b\sin(fx+e)+a}}{(c\sin(fx+e)+c)\sqrt{g\sin(fx+e)}}\,dx$$

input
```
integrate((a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e))/(g*sin(f*x+e))^(1/2),x,
algorithm="giac")
```

output
```
integrate(sqrt(b*sin(f*x + e) + a)/((c*sin(f*x + e) + c)*sqrt(g*sin(f*x +
e))), x)
```

3.32.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c+c\sin(e+fx))}\,dx=\int \frac{\sqrt{a+b\,\sin\left(e+f\,x\right)}}{\sqrt{g\,\sin\left(e+f\,x\right)}\,\left(c+c\,\sin\left(e+f\,x\right)\right)}\,dx$$

input
```
int((a + b*sin(e + f*x))^(1/2)/((g*sin(e + f*x))^(1/2)*(c + c*sin(e + f*x)
)),x)
```

output
```
int((a + b*sin(e + f*x))^(1/2)/((g*sin(e + f*x))^(1/2)*(c + c*sin(e + f*x)
)), x)
```

3.33 $\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}\,dx$

3.33.1 Optimal result

Integrand size = 39, antiderivative size = 252

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}\,dx = \frac{gE\left(\arcsin\left(\frac{\cos(e+fx)}{1+\sin(e+fx)}\right)|-\frac{a-b}{a+b}\right)\sqrt{\frac{\sin(e+fx)}{1+\sin(e+fx)}}\sqrt{a+b\sin(e+fx)}}{(a-b)cf\sqrt{g\sin(e+fx)}\sqrt{\frac{a+b\sin(e+fx)}{(a+b)(1+\sin(e+fx))}}} - \frac{2\sqrt{a+b}\sqrt{g}\sqrt{\frac{a(1-\csc(e+fx))}{a+b}}\sqrt{\frac{a(1+\csc(e+fx))}{a-b}}\operatorname{EllipticF}\left(\arcsin\left(\frac{\sqrt{g}\sqrt{a+b\sin(e+fx)}}{\sqrt{a+b}\sqrt{g\sin(e+fx)}}\right),-\frac{a+b}{a-b}\right)\tan(e+fx)}{(a-b)cf}$$

output
```
g*EllipticE(cos(f*x+e)/(1+sin(f*x+e)),((-a+b)/(a+b))^(1/2))*(sin(f*x+e)/(1
+sin(f*x+e)))^(1/2)*(a+b*sin(f*x+e))^(1/2)/(a-b)/c/f/(g*sin(f*x+e))^(1/2)/
((a+b*sin(f*x+e))/(a+b)/(1+sin(f*x+e)))^(1/2)-2*EllipticF(g^(1/2)*(a+b*sin
(f*x+e))^(1/2)/(a+b)^(1/2)/(g*sin(f*x+e))^(1/2),((-a-b)/(a-b))^(1/2))*(a+b
)^(1/2)*g^(1/2)*(a*(1-csc(f*x+e))/(a+b))^(1/2)*(a*(1+csc(f*x+e))/(a-b))^(1
/2)*tan(f*x+e)/(a-b)/c/f
```

3.33.2 Mathematica [B] (warning: unable to verify)

Leaf count is larger than twice the leaf count of optimal. 4464 vs. $2(252) = 504$.

Time = 28.85 (sec) , antiderivative size = 4464, normalized size of antiderivative = 17.71

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}\,dx = \text{Result too large to show}$$

input
```
Integrate[Sqrt[g*Sin[e + f*x]]/(Sqrt[a + b*Sin[e + f*x]]*(c + c*Sin[e + f*
x])),x]
```

output
```
(2*Sin[(e + f*x)/2]*(Cos[(e + f*x)/2] + Sin[(e + f*x)/2])*Sqrt[g*Sin[e + f
*x]]*Sqrt[a + b*Sin[e + f*x]])/((a - b)*f*(c + c*Sin[e + f*x])) + (Cot[(e
+ f*x)/2]*(Cos[(e + f*x)/2] + Sin[(e + f*x)/2])^2*Sqrt[g*Sin[e + f*x]]*Sqr
t[a + b*Sin[e + f*x]]*(-1/2*(a*Sqrt[Sin[e + f*x]])/((a - b)*Sqrt[a + b*Sin
[e + f*x]]) - (b*Sqrt[Sin[e + f*x]])/(2*(a - b)*Sqrt[a + b*Sin[e + f*x]])
- (b*Cos[(3*(e + f*x))/2]*Sec[(e + f*x)/2]*Sqrt[Sin[e + f*x]])/(2*(a - b)*
Sqrt[a + b*Sin[e + f*x]]) - (b*Sec[(e + f*x)/2]*Sqrt[Sin[e + f*x]]*Sin[(3*
(e + f*x))/2])/(2*(a - b)*Sqrt[a + b*Sin[e + f*x]]) + (a*Sqrt[Sin[e + f*x]
]*Tan[(e + f*x)/2])/(2*(a - b)*Sqrt[a + b*Sin[e + f*x]]) - (b*Sqrt[Sin[e +
 f*x]]*Tan[(e + f*x)/2])/(2*(a - b)*Sqrt[a + b*Sin[e + f*x]]))*(-2*Tan[(e
+ f*x)/2]*(1 + Tan[(e + f*x)/2]) + (2*Sqrt[-a^2 + b^2]*Sqrt[(a*Sec[(e + f*
x)/2]^2*(a + b*Sin[e + f*x]))/(a^2 - b^2)]*(-(EllipticE[ArcSin[Sqrt[(-b +
Sqrt[-a^2 + b^2] - a*Tan[(e + f*x)/2])/Sqrt[-a^2 + b^2]]/Sqrt[2]], (2*Sqrt
[-a^2 + b^2])/(-b + Sqrt[-a^2 + b^2])]*Tan[(e + f*x)/2]) + EllipticF[ArcSi
n[Sqrt[(b + Sqrt[-a^2 + b^2] + a*Tan[(e + f*x)/2])/Sqrt[-a^2 + b^2]]/Sqrt[
2]], (2*Sqrt[-a^2 + b^2])/(b + Sqrt[-a^2 + b^2])]*Sqrt[(a*Tan[(e + f*x)/2]
)/(-b + Sqrt[-a^2 + b^2])]*Sqrt[-((a*Tan[(e + f*x)/2])/(b + Sqrt[-a^2 + b^
2]))]))/((a + b*Sin[e + f*x])*Sqrt[(a*Tan[(e + f*x)/2])/(-b + Sqrt[-a^2 +
b^2])]))))/(2*(a - b)*f*(c + c*Sin[e + f*x])*((b*Cos[e + f*x]*Cot[(e + f*x)
/2]*Sqrt[Sin[e + f*x]]*(-2*Tan[(e + f*x)/2]*(1 + Tan[(e + f*x)/2]) + (2...
```

3.33.3 Rubi [A] (verified)

Time = 0.95 (sec) , antiderivative size = 252, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.128$, Rules used = {3042, 3415, 3042, 3295, 3411}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

3.33. $\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}\,dx$

$$\int \frac{\sqrt{g\sin(e+fx)}}{(c\sin(e+fx)+c)\sqrt{a+b\sin(e+fx)}}dx$$

$\downarrow$ 3042

$$\int \frac{\sqrt{g\sin(e+fx)}}{(c\sin(e+fx)+c)\sqrt{a+b\sin(e+fx)}}dx$$

$\downarrow$ 3415

$$\frac{ag\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}dx}{c(a-b)}-\frac{g\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(\sin(e+fx)c+c)}dx}{a-b}$$

$\downarrow$ 3042

$$\frac{ag\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}dx}{c(a-b)}-\frac{g\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(\sin(e+fx)c+c)}dx}{a-b}$$

$\downarrow$ 3295

$$-\frac{g\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(\sin(e+fx)c+c)}dx}{a-b}-\frac{2\sqrt{g}\sqrt{a+b}\tan(e+fx)\sqrt{\frac{a(1-\csc(e+fx))}{a+b}}\sqrt{\frac{a(\csc(e+fx)+1)}{a-b}}\,\text{EllipticF}\left(\arcsin\left(\frac{\sqrt{g}\sqrt{a+b\sin(e+fx)}}{\sqrt{a+b}\sqrt{g\sin(e+fx)}}\right),-\frac{a+b}{a-b}\right)}{cf(a-b)}$$

$\downarrow$ 3411

$$\frac{g\sqrt{\frac{\sin(e+fx)}{\sin(e+fx)+1}}\sqrt{a+b\sin(e+fx)}E\left(\arcsin\left(\frac{\cos(e+fx)}{\sin(e+fx)+1}\right)|-\frac{a-b}{a+b}\right)}{cf(a-b)\sqrt{g\sin(e+fx)}\sqrt{\frac{a+b\sin(e+fx)}{(a+b)(\sin(e+fx)+1)}}}-\frac{2\sqrt{g}\sqrt{a+b}\tan(e+fx)\sqrt{\frac{a(1-\csc(e+fx))}{a+b}}\sqrt{\frac{a(\csc(e+fx)+1)}{a-b}}\,\text{EllipticF}\left(\arcsin\left(\frac{\sqrt{g}\sqrt{a+b\sin(e+fx)}}{\sqrt{a+b}\sqrt{g\sin(e+fx)}}\right),-\frac{a+b}{a-b}\right)}{cf(a-b)}$$

input
```
Int[Sqrt[g*Sin[e + f*x]]/(Sqrt[a + b*Sin[e + f*x]]*(c + c*Sin[e + f*x])),x
]
```

output
```
(g*EllipticE[ArcSin[Cos[e + f*x]/(1 + Sin[e + f*x])], -((a - b)/(a + b))]*
Sqrt[Sin[e + f*x]/(1 + Sin[e + f*x])]*Sqrt[a + b*Sin[e + f*x]])/((a - b)*c
*f*Sqrt[g*Sin[e + f*x]]*Sqrt[(a + b*Sin[e + f*x])/((a + b)*(1 + Sin[e + f*
x]))]) - (2*Sqrt[a + b]*Sqrt[g]*Sqrt[(a*(1 - Csc[e + f*x]))/(a + b)]*Sqrt[
(a*(1 + Csc[e + f*x]))/(a - b)]*EllipticF[ArcSin[(Sqrt[g]*Sqrt[a + b*Sin[e
 + f*x]])/(Sqrt[a + b]*Sqrt[g*Sin[e + f*x]])], -((a + b)/(a - b))]*Tan[e +
 f*x])/((a - b)*c*f)
```

3.33. $\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}dx$

3.33.3.1 Defintions of rubi rules used

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3295
```
Int[1/(Sqrt[(d_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(a_) + (b_.)*sin[(e_.) + (f
_.)*(x_)]]), x_Symbol] :> Simp[-2*(Tan[e + f*x]/(a*f))*Rt[(a + b)/d, 2]*Sqr
t[a*((1 - Csc[e + f*x])/(a + b))]*Sqrt[a*((1 + Csc[e + f*x])/(a - b))]*Elli
pticF[ArcSin[Sqrt[a + b*Sin[e + f*x]]/Sqrt[d*Sin[e + f*x]]/Rt[(a + b)/d, 2]
], -(a + b)/(a - b)], x] /; FreeQ[{a, b, d, e, f}, x] && NeQ[a^2 - b^2, 0]
&& PosQ[(a + b)/d]
```

rule 3411
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(g_.)*sin[(e_.) + (f_.
)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[(-Sqrt[
a + b*Sin[e + f*x]])*(Sqrt[d*(Sin[e + f*x]/(c + d*Sin[e + f*x]))]/(d*f*Sqrt
[g*Sin[e + f*x]]*Sqrt[c^2*((a + b*Sin[e + f*x])/((a*c + b*d)*(c + d*Sin[e +
 f*x])))]))*EllipticE[ArcSin[c*(Cos[e + f*x]/(c + d*Sin[e + f*x]))], (b*c -
 a*d)/(b*c + a*d)], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d,
 0] && NeQ[a^2 - b^2, 0] && EqQ[c^2 - d^2, 0]
```

rule 3415
```
Int[Sqrt[(g_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.
)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[(-a)*(g
/(b*c - a*d))   Int[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]), x],
x] + Simp[c*(g/(b*c - a*d))   Int[Sqrt[a + b*Sin[e + f*x]]/(Sqrt[g*Sin[e +
f*x]]*(c + d*Sin[e + f*x])), x], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && N
eQ[b*c - a*d, 0] && (EqQ[a^2 - b^2, 0] || EqQ[c^2 - d^2, 0])
```

3.33.4 Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 3056 vs. $2(232) = 464$.

Time = 2.48 (sec) , antiderivative size = 3057, normalized size of antiderivative = 12.13

method	result	size
default	Expression too large to display	3057

input
```
int((g*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e))/(a+b*sin(f*x+e))^(1/2),x,method=
_RETURNVERBOSE)
```

output
```
1/2/c/f*(g/((1-cos(f*x+e))^2*csc(f*x+e)^2+1)*(csc(f*x+e)-cot(f*x+e)))^(1/2
)*((1-cos(f*x+e))^2*csc(f*x+e)^2+1)*((a*(1-cos(f*x+e))^2*csc(f*x+e)^2+2*b*
(csc(f*x+e)-cot(f*x+e))+a)/((1-cos(f*x+e))^2*csc(f*x+e)^2+1))^(1/2)*((1/(b
+(-a^2+b^2)^(1/2))*(a*(csc(f*x+e)-cot(f*x+e))+(-a^2+b^2)^(1/2)+b))^(1/2)*(
1/(-a^2+b^2)^(1/2)*(-a*(csc(f*x+e)-cot(f*x+e))+(-a^2+b^2)^(1/2)-b))^(1/2)*
(-a/(b+(-a^2+b^2)^(1/2))*(csc(f*x+e)-cot(f*x+e)))^(1/2)*EllipticF((1/(b+(-
a^2+b^2)^(1/2))*(a*(csc(f*x+e)-cot(f*x+e))+(-a^2+b^2)^(1/2)+b))^(1/2),1/2*
2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1/2))*(-a^2+b^2)^(1/2)*2^
(1/2)*a*(csc(f*x+e)-cot(f*x+e))-2*(1/(b+(-a^2+b^2)^(1/2))*(a*(csc(f*x+e)-c
ot(f*x+e))+(-a^2+b^2)^(1/2)+b))^(1/2)*(1/(-a^2+b^2)^(1/2)*(-a*(csc(f*x+e)-
cot(f*x+e))+(-a^2+b^2)^(1/2)-b))^(1/2)*(-a/(b+(-a^2+b^2)^(1/2))*(csc(f*x+e
)-cot(f*x+e)))^(1/2)*EllipticE((1/(b+(-a^2+b^2)^(1/2))*(a*(csc(f*x+e)-cot(
f*x+e))+(-a^2+b^2)^(1/2)+b))^(1/2),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2
+b^2)^(1/2))^(1/2))*(-a^2+b^2)^(1/2)*2^(1/2)*b*(csc(f*x+e)-cot(f*x+e))-(1/
(b+(-a^2+b^2)^(1/2))*(a*(csc(f*x+e)-cot(f*x+e))+(-a^2+b^2)^(1/2)+b))^(1/2)
*(1/(-a^2+b^2)^(1/2)*(-a*(csc(f*x+e)-cot(f*x+e))+(-a^2+b^2)^(1/2)-b))^(1/2
)*(-a/(b+(-a^2+b^2)^(1/2))*(csc(f*x+e)-cot(f*x+e)))^(1/2)*EllipticF((1/(b+
(-a^2+b^2)^(1/2))*(a*(csc(f*x+e)-cot(f*x+e))+(-a^2+b^2)^(1/2)+b))^(1/2),1/
2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1/2))*2^(1/2)*a^2*(csc(
f*x+e)-cot(f*x+e))+(1/(b+(-a^2+b^2)^(1/2))*(a*(csc(f*x+e)-cot(f*x+e))+(...
```

3.33.5 Fricas [F]

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}\,dx=\int \frac{\sqrt{g\sin\left(fx+e\right)}}{\sqrt{b\sin\left(fx+e\right)+a}\left(c\sin\left(fx+e\right)+c\right)}\,dx$$

input
```
integrate((g*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e))/(a+b*sin(f*x+e))^(1/2),x,
algorithm="fricas")
```

output
```
integral(-sqrt(b*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))/(b*c*cos(f*x + e)^
2 - (a + b)*c*sin(f*x + e) - (a + b)*c), x)
```

3.33.6 Sympy [F]

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}\,dx = \frac{\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}\sin(e+fx)+\sqrt{a+b\sin(e+fx)}}\,dx}{c}$$

input
```
integrate((g*sin(f*x+e))**(1/2)/(c+c*sin(f*x+e))/(a+b*sin(f*x+e))**(1/2),x
)
```

output
```
Integral(sqrt(g*sin(e + f*x))/(sqrt(a + b*sin(e + f*x))*sin(e + f*x) + sqr
t(a + b*sin(e + f*x))), x)/c
```

3.33.7 Maxima [F]

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}\,dx = \int \frac{\sqrt{g\sin(fx+e)}}{\sqrt{b\sin(fx+e)+a}(c\sin(fx+e)+c)}\,dx$$

input
```
integrate((g*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e))/(a+b*sin(f*x+e))^(1/2),x,
algorithm="maxima")
```

output
```
integrate(sqrt(g*sin(f*x + e))/(sqrt(b*sin(f*x + e) + a)*(c*sin(f*x + e) +
 c)), x)
```

3.33.8 Giac [F]

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}\,dx = \int \frac{\sqrt{g\sin(fx+e)}}{\sqrt{b\sin(fx+e)+a}(c\sin(fx+e)+c)}\,dx$$

input
```
integrate((g*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e))/(a+b*sin(f*x+e))^(1/2),x,
algorithm="giac")
```

output
```
integrate(sqrt(g*sin(f*x + e))/(sqrt(b*sin(f*x + e) + a)*(c*sin(f*x + e) +
 c)), x)
```

3.33.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}\,dx$$
$$=\int \frac{\sqrt{g\,\sin\left(e+f\,x\right)}}{\sqrt{a+b\,\sin\left(e+f\,x\right)}\,\left(c+c\,\sin\left(e+f\,x\right)\right)}\,dx$$

input
```
int((g*sin(e + f*x))^(1/2)/((a + b*sin(e + f*x))^(1/2)*(c + c*sin(e + f*x)
)),x)
```

output
```
int((g*sin(e + f*x))^(1/2)/((a + b*sin(e + f*x))^(1/2)*(c + c*sin(e + f*x)
)), x)
```

3.34 $\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}\,dx$

3.34.1 Optimal result

Integrand size = 39, antiderivative size = 256

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}\,dx$$
$$= -\frac{E\left(\arcsin\left(\frac{\cos(e+fx)}{1+\sin(e+fx)}\right)|-\frac{a-b}{a+b}\right)\sqrt{\frac{\sin(e+fx)}{1+\sin(e+fx)}}\sqrt{a+b\sin(e+fx)}}{(a-b)cf\sqrt{g\sin(e+fx)}\sqrt{\frac{a+b\sin(e+fx)}{(a+b)(1+\sin(e+fx))}}}$$
$$+\frac{2b\sqrt{a+b}\sqrt{\frac{a(1-\csc(e+fx))}{a+b}}\sqrt{\frac{a(1+\csc(e+fx))}{a-b}}\,\text{EllipticF}\left(\arcsin\left(\frac{\sqrt{g}\sqrt{a+b\sin(e+fx)}}{\sqrt{a+b}\sqrt{g\sin(e+fx)}}\right),-\frac{a+b}{a-b}\right)\tan(e+fx)}{a(a-b)cf\sqrt{g}}$$

output
```
-EllipticE(cos(f*x+e)/(1+sin(f*x+e)),((-a+b)/(a+b))^(1/2))*(sin(f*x+e)/(1+
sin(f*x+e)))^(1/2)*(a+b*sin(f*x+e))^(1/2)/(a-b)/c/f/(g*sin(f*x+e))^(1/2)/(
(a+b*sin(f*x+e))/(a+b)/(1+sin(f*x+e)))^(1/2)+2*b*EllipticF(g^(1/2)*(a+b*si
n(f*x+e))^(1/2)/(a+b)^(1/2)/(g*sin(f*x+e))^(1/2),((-a-b)/(a-b))^(1/2))*(a+
b)^(1/2)*(a*(1-csc(f*x+e))/(a+b))^(1/2)*(a*(1+csc(f*x+e))/(a-b))^(1/2)*tan
(f*x+e)/a/(a-b)/c/f/g^(1/2)
```

3.34.2 Mathematica [C] (warning: unable to verify)

Result contains complex when optimal does not.

Time = 19.75 (sec) , antiderivative size = 1667, normalized size of antiderivative = 6.51

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}\,dx = \text{Too large to display}$$

input
```
Integrate[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]*(c + c*Sin[e +
f*x])),x]
```

output
```
(-2*Sin[(e + f*x)/2]*(Cos[(e + f*x)/2] + Sin[(e + f*x)/2])*Sin[e + f*x]*Sq
rt[a + b*Sin[e + f*x]])/((a - b)*f*Sqrt[g*Sin[e + f*x]]*(c + c*Sin[e + f*x
])) + ((Cos[(e + f*x)/2] + Sin[(e + f*x)/2])^2*Sqrt[Sin[e + f*x]]*((4*a*(a
 - b)*Sqrt[((a + b)*Cot[(-e + Pi/2 - f*x)/2]^2)/(-a + b)]*EllipticF[ArcSin
[Sqrt[(Csc[(-e + Pi/2 - f*x)/2]^2*(a + b*Sin[e + f*x]))/a]/Sqrt[2]], (-2*a
)/(-a + b)]*Sec[e + f*x]*Sin[(-e + Pi/2 - f*x)/2]^4*Sqrt[-(((a + b)*Csc[(-
e + Pi/2 - f*x)/2]^2*Sin[e + f*x])/a)]*Sqrt[(Csc[(-e + Pi/2 - f*x)/2]^2*(a
 + b*Sin[e + f*x]))/a])/((a + b)*Sqrt[Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x
]]) + (2*a*ArcTanh[(Sqrt[b]*Sqrt[Sin[e + f*x]])/Sqrt[a + b*Sin[e + f*x]]]*
Cos[e + f*x]^2)/(Sqrt[b]*(1 - Sin[e + f*x]^2)) + 4*a^2*((Sqrt[((a + b)*Cot
[(-e + Pi/2 - f*x)/2]^2)/(-a + b)]*EllipticF[ArcSin[Sqrt[(Csc[(-e + Pi/2 -
 f*x)/2]^2*(a + b*Sin[e + f*x]))/a]/Sqrt[2]], (-2*a)/(-a + b)]*Sec[e + f*x
]*Sin[(-e + Pi/2 - f*x)/2]^4*Sqrt[-(((a + b)*Csc[(-e + Pi/2 - f*x)/2]^2*Si
n[e + f*x])/a)]*Sqrt[(Csc[(-e + Pi/2 - f*x)/2]^2*(a + b*Sin[e + f*x]))/a])
/((a + b)*Sqrt[Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]) - (Sqrt[((a + b)*Co
t[(-e + Pi/2 - f*x)/2]^2)/(-a + b)]*EllipticPi[-(a/b), ArcSin[Sqrt[(Csc[(-
e + Pi/2 - f*x)/2]^2*(a + b*Sin[e + f*x]))/a]/Sqrt[2]], (-2*a)/(-a + b)]*S
ec[e + f*x]*Sin[(-e + Pi/2 - f*x)/2]^4*Sqrt[-(((a + b)*Csc[(-e + Pi/2 - f*
x)/2]^2*Sin[e + f*x])/a)]*Sqrt[(Csc[(-e + Pi/2 - f*x)/2]^2*(a + b*Sin[e +
f*x]))/a])/(b*Sqrt[Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]])) - 2*b*((Cos...
```

3.34.3 Rubi [A] (verified)

Time = 0.97 (sec) , antiderivative size = 256, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.128$, Rules used = {3042, 3417, 3042, 3295, 3411}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{1}{(c\sin(e+fx)+c)\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}dx$$

$$\Big\downarrow \text{ 3042}$$

$$\int \frac{1}{(c\sin(e+fx)+c)\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}dx$$

$$\Big\downarrow \text{ 3417}$$

$$\frac{\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(\sin(e+fx)c+c)}dx}{a-b}-\frac{b\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}dx}{c(a-b)}$$

$$\Big\downarrow \text{ 3042}$$

$$\frac{\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(\sin(e+fx)c+c)}dx}{a-b}-\frac{b\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}dx}{c(a-b)}$$

$$\Big\downarrow \text{ 3295}$$

$$\frac{\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(\sin(e+fx)c+c)}dx}{a-b}+\frac{2b\sqrt{a+b}\tan(e+fx)\sqrt{\frac{a(1-\csc(e+fx))}{a+b}}\sqrt{\frac{a(\csc(e+fx)+1)}{a-b}}\text{EllipticF}\left(\arcsin\left(\frac{\sqrt{g}\sqrt{a+b\sin(e+fx)}}{\sqrt{a+b}\sqrt{g\sin(e+fx)}}\right),-\frac{a+b}{a-b}\right)}{acf\sqrt{g}(a-b)}$$

$$\Big\downarrow \text{ 3411}$$

$$\frac{2b\sqrt{a+b}\tan(e+fx)\sqrt{\frac{a(1-\csc(e+fx))}{a+b}}\sqrt{\frac{a(\csc(e+fx)+1)}{a-b}}\text{EllipticF}\left(\arcsin\left(\frac{\sqrt{g}\sqrt{a+b\sin(e+fx)}}{\sqrt{a+b}\sqrt{g\sin(e+fx)}}\right),-\frac{a+b}{a-b}\right)}{acf\sqrt{g}(a-b)}-\frac{\sqrt{\frac{\sin(e+fx)}{\sin(e+fx)+1}}\sqrt{a+b\sin(e+fx)}E\left(\arcsin\left(\frac{\cos(e+fx)}{\sin(e+fx)+1}\right)|-\frac{a-b}{a+b}\right)}{cf(a-b)\sqrt{g\sin(e+fx)}\sqrt{\frac{a+b\sin(e+fx)}{(a+b)(\sin(e+fx)+1)}}}$$

input
```
Int[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]*(c + c*Sin[e + f*x]))
,x]
```

3.34. $\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}dx$

output
```
-((EllipticE[ArcSin[Cos[e + f*x]/(1 + Sin[e + f*x])], -((a - b)/(a + b))]*
Sqrt[Sin[e + f*x]/(1 + Sin[e + f*x])]*Sqrt[a + b*Sin[e + f*x]])/((a - b)*c
*f*Sqrt[g*Sin[e + f*x]]*Sqrt[(a + b*Sin[e + f*x])/((a + b)*(1 + Sin[e + f*
x]))])) + (2*b*Sqrt[a + b]*Sqrt[(a*(1 - Csc[e + f*x]))/(a + b)]*Sqrt[(a*(1
 + Csc[e + f*x]))/(a - b)]*EllipticF[ArcSin[(Sqrt[g]*Sqrt[a + b*Sin[e + f*
x]])/(Sqrt[a + b]*Sqrt[g*Sin[e + f*x]])], -((a + b)/(a - b))]*Tan[e + f*x]
)/(a*(a - b)*c*f*Sqrt[g])
```

3.34.3.1 Defintions of rubi rules used

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3295
```
Int[1/(Sqrt[(d_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(a_) + (b_.)*sin[(e_.) + (f
_.)*(x_)]]), x_Symbol] :> Simp[-2*(Tan[e + f*x]/(a*f))*Rt[(a + b)/d, 2]*Sqr
t[a*((1 - Csc[e + f*x])/(a + b))]*Sqrt[a*((1 + Csc[e + f*x])/(a - b))]*Elli
pticF[ArcSin[Sqrt[a + b*Sin[e + f*x]]/Sqrt[d*Sin[e + f*x]]/Rt[(a + b)/d, 2]
], -(a + b)/(a - b)], x] /; FreeQ[{a, b, d, e, f}, x] && NeQ[a^2 - b^2, 0]
&& PosQ[(a + b)/d]
```

rule 3411
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(g_.)*sin[(e_.) + (f_.
)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[(-Sqrt[
a + b*Sin[e + f*x]])*(Sqrt[d*(Sin[e + f*x]/(c + d*Sin[e + f*x]))]/(d*f*Sqrt
[g*Sin[e + f*x]]*Sqrt[c^2*((a + b*Sin[e + f*x])/((a*c + b*d)*(c + d*Sin[e +
 f*x])))]))*EllipticE[ArcSin[c*(Cos[e + f*x]/(c + d*Sin[e + f*x]))], (b*c -
 a*d)/(b*c + a*d)], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d,
 0] && NeQ[a^2 - b^2, 0] && EqQ[c^2 - d^2, 0]
```

rule 3417
```
Int[1/(Sqrt[(g_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(a_) + (b_.)*sin[(e_.) + (f
_.)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[b/(b*
c - a*d)   Int[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]), x], x] -
Simp[d/(b*c - a*d)   Int[Sqrt[a + b*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c
+ d*Sin[e + f*x])), x], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c -
a*d, 0] && (EqQ[a^2 - b^2, 0] || EqQ[c^2 - d^2, 0])
```

3.34.4 Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 3937 vs. $2(236) = 472$.

Time = 2.61 (sec) , antiderivative size = 3938, normalized size of antiderivative = 15.38

method	result	size
default	Expression too large to display	3938

input
```
int(1/(c+c*sin(f*x+e))/(g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2),x,metho
d=_RETURNVERBOSE)
```

output
```
1/2/c/f/(g/((1-cos(f*x+e))^2*csc(f*x+e)^2+1)*(csc(f*x+e)-cot(f*x+e)))^(1/2
)*((a*(1-cos(f*x+e))^2*csc(f*x+e)^2+2*b*(csc(f*x+e)-cot(f*x+e))+a)/((1-cos
(f*x+e))^2*csc(f*x+e)^2+1))^(1/2)*((1/(b+(-a^2+b^2)^(1/2))*(a*(csc(f*x+e)-
cot(f*x+e))+(-a^2+b^2)^(1/2)+b))^(1/2)*(1/(-a^2+b^2)^(1/2)*(-a*(csc(f*x+e)
-cot(f*x+e))+(-a^2+b^2)^(1/2)-b))^(1/2)*(-a/(b+(-a^2+b^2)^(1/2))*(csc(f*x+
e)-cot(f*x+e)))^(1/2)*EllipticF((1/(b+(-a^2+b^2)^(1/2))*(a*(csc(f*x+e)-cot
(f*x+e))+(-a^2+b^2)^(1/2)+b))^(1/2),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^
2+b^2)^(1/2))^(1/2))*(-a^2+b^2)^(1/2)*2^(1/2)*a*(csc(f*x+e)-cot(f*x+e))-2*
2^(1/2)*(-a^2+b^2)^(1/2)*(1/(b+(-a^2+b^2)^(1/2))*(a*(csc(f*x+e)-cot(f*x+e)
)+(-a^2+b^2)^(1/2)+b))^(1/2)*(1/(-a^2+b^2)^(1/2)*(-a*(csc(f*x+e)-cot(f*x+e
))+(-a^2+b^2)^(1/2)-b))^(1/2)*(-a/(b+(-a^2+b^2)^(1/2))*(csc(f*x+e)-cot(f*x
+e)))^(1/2)*EllipticF((1/(b+(-a^2+b^2)^(1/2))*(a*(csc(f*x+e)-cot(f*x+e))+(
-a^2+b^2)^(1/2)+b))^(1/2),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/
2))^(1/2))*b*(csc(f*x+e)-cot(f*x+e))+2*(1/(b+(-a^2+b^2)^(1/2))*(a*(csc(f*x
+e)-cot(f*x+e))+(-a^2+b^2)^(1/2)+b))^(1/2)*(1/(-a^2+b^2)^(1/2)*(-a*(csc(f*
x+e)-cot(f*x+e))+(-a^2+b^2)^(1/2)-b))^(1/2)*(-a/(b+(-a^2+b^2)^(1/2))*(csc(
f*x+e)-cot(f*x+e)))^(1/2)*EllipticE((1/(b+(-a^2+b^2)^(1/2))*(a*(csc(f*x+e)
-cot(f*x+e))+(-a^2+b^2)^(1/2)+b))^(1/2),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/
(-a^2+b^2)^(1/2))^(1/2))*(-a^2+b^2)^(1/2)*2^(1/2)*b*(csc(f*x+e)-cot(f*x+e)
)+(1/(b+(-a^2+b^2)^(1/2))*(a*(csc(f*x+e)-cot(f*x+e))+(-a^2+b^2)^(1/2)+b...
```

3.34.5 Fricas [F]

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}\,dx$$
$$= \int \frac{1}{\sqrt{b\sin(fx+e)+a}(c\sin(fx+e)+c)\sqrt{g\sin(fx+e)}}\,dx$$

input
```
integrate(1/(c+c*sin(f*x+e))/(g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2),x
, algorithm="fricas")
```

output
```
integral(-sqrt(b*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))/((a + b)*c*g*cos(f
*x + e)^2 - (a + b)*c*g + (b*c*g*cos(f*x + e)^2 - (a + b)*c*g)*sin(f*x + e
)), x)
```

3.34.6 Sympy [F]

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}\,dx = \frac{\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}\sin(e+fx)+\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}\,dx}{c}$$

input
```
integrate(1/(c+c*sin(f*x+e))/(g*sin(f*x+e))**(1/2)/(a+b*sin(f*x+e))**(1/2)
,x)
```

output
```
Integral(1/(sqrt(g*sin(e + f*x))*sqrt(a + b*sin(e + f*x))*sin(e + f*x) + s
qrt(g*sin(e + f*x))*sqrt(a + b*sin(e + f*x))), x)/c
```

3.34.7 Maxima [F]

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}\,dx = \int \frac{1}{\sqrt{b\sin(fx+e)+a}(c\sin(fx+e)+c)\sqrt{g\sin(fx+e)}}\,dx$$

input
```
integrate(1/(c+c*sin(f*x+e))/(g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2),x
, algorithm="maxima")
```

output
```
integrate(1/(sqrt(b*sin(f*x + e) + a)*(c*sin(f*x + e) + c)*sqrt(g*sin(f*x
+ e))), x)
```

3.34.8 Giac [F]

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}\,dx$$
$$=\int \frac{1}{\sqrt{b\sin{(fx+e)}+a}(c\sin{(fx+e)}+c)\sqrt{g\sin{(fx+e)}}}\,dx$$

input
```
integrate(1/(c+c*sin(f*x+e))/(g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2),x
, algorithm="giac")
```

output
```
integrate(1/(sqrt(b*sin(f*x + e) + a)*(c*sin(f*x + e) + c)*sqrt(g*sin(f*x
+ e))), x)
```

3.34.9 Mupad [F(-1)]

Timed out.

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}(c+c\sin(e+fx))}\,dx$$
$$=\int \frac{1}{\sqrt{g\,\sin{(e+f\,x)}}\,\sqrt{a+b\,\sin{(e+f\,x)}}\,(c+c\,\sin{(e+f\,x)})}\,dx$$

input
```
int(1/((g*sin(e + f*x))^(1/2)*(a + b*sin(e + f*x))^(1/2)*(c + c*sin(e + f*
x))),x)
```

output
```
int(1/((g*sin(e + f*x))^(1/2)*(a + b*sin(e + f*x))^(1/2)*(c + c*sin(e + f*
x))), x)
```

3.35 $\int \csc(e+fx)\sqrt{a+a\sin(e+fx)}\sqrt{c+d\sin(e+fx)}\,dx$

3.35.1 Optimal result

Integrand size = 35, antiderivative size = 123

$$\int \csc(e+fx)\sqrt{a+a\sin(e+fx)}\sqrt{c+d\sin(e+fx)}\,dx = -\frac{2\sqrt{a}\sqrt{d}\arctan\left(\frac{\sqrt{a}\sqrt{d}\cos(e+fx)}{\sqrt{a+a\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\right)}{f} - \frac{2\sqrt{a}\sqrt{c}\,\mathrm{arctanh}\left(\frac{\sqrt{a}\sqrt{c}\cos(e+fx)}{\sqrt{a+a\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\right)}{f}$$

output
```
-2*arctanh(cos(f*x+e)*a^(1/2)*c^(1/2)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+
e))^(1/2))*a^(1/2)*c^(1/2)/f-2*arctan(cos(f*x+e)*a^(1/2)*d^(1/2)/(a+a*sin(
f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2))*a^(1/2)*d^(1/2)/f
```

3.35.2 Mathematica [C] (warning: unable to verify)

Result contains complex when optimal does not.

Time = 2.90 (sec) , antiderivative size = 567, normalized size of antiderivative = 4.61

$$\int \csc(e+fx)\sqrt{a+a\sin(e+fx)}\sqrt{c+d\sin(e+fx)}\,dx = -\frac{\left(\sqrt{c}\log\left(\frac{\left(\frac{1}{2}+\frac{i}{2}\right)e^{-\frac{ie}{2}}\left(-\sqrt{2}c\left(-1+e^{i(e+fx)}\right)-i\sqrt{2}d\left(1+e^{i(e+fx)}\right)+2i\sqrt{c}\sqrt{2ce^{i(e+fx)}-id\left(-1+e^{2i(e+fx)}\right)}\right)f}{c^{3/2}\left(1+e^{i(e+fx)}\right)}\right)+\sqrt{c}\log\left(\frac{\left(\frac{1}{2}+\frac{i}{2}\right)\cdots}{\cdots}\right.\cdots}{\cdots}$$

input
```
Integrate[Csc[e + f*x]*Sqrt[a + a*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]],x
]
```

output
```
-(((Sqrt[c]*Log[((1/2 + I/2)*(-(Sqrt[2]*c*(-1 + E^(I*(e + f*x)))) - I*Sqrt
[2]*d*(1 + E^(I*(e + f*x))) + (2*I)*Sqrt[c]*Sqrt[2*c*E^(I*(e + f*x)) - I*d
*(-1 + E^((2*I)*(e + f*x)))])*f)/(c^(3/2)*E^((I/2)*e)*(1 + E^(I*(e + f*x))
))] + Sqrt[c]*Log[((1/2 + I/2)*((-I)*Sqrt[2]*d*(-1 + E^(I*(e + f*x))) + Sq
rt[2]*c*(1 + E^(I*(e + f*x))) + 2*Sqrt[c]*Sqrt[2*c*E^(I*(e + f*x)) - I*d*(
-1 + E^((2*I)*(e + f*x)))])*f)/(c^(3/2)*E^((I/2)*e)*(-1 + E^(I*(e + f*x)))
)] - I*Sqrt[d]*(Log[(2*((-1)^(3/4)*d + (-1)^(1/4)*c*E^(I*(e + f*x)) + I*Sq
rt[d]*Sqrt[2*c*E^(I*(e + f*x)) - I*d*(-1 + E^((2*I)*(e + f*x)))])*f)/(d^(3
/2)*E^((I/2)*(e + 2*f*x)))] - Log[((1 + I)*Sqrt[2]*(c - I*d*Cos[e + f*x] +
 d*Sin[e + f*x] + (1 - I)*Sqrt[d]*Sqrt[(Cos[e + f*x] + I*Sin[e + f*x])*(c
+ d*Sin[e + f*x])]))/Sqrt[d]]))*(Cos[(e + f*x)/2] + I*Sin[(e + f*x)/2])*Sq
rt[a*(1 + Sin[e + f*x])]*Sqrt[c + d*Sin[e + f*x]])/(f*(Cos[(e + f*x)/2] +
Sin[(e + f*x)/2])*Sqrt[(Cos[e + f*x] + I*Sin[e + f*x])*(c + d*Sin[e + f*x]
)]))
```

3.35.3 Rubi [A] (verified)

Time = 0.88 (sec) , antiderivative size = 123, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}} = 0.200$, Rules used = {3042, 3428, 3042, 3254, 218, 3422, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \csc(e+fx)\sqrt{a\sin(e+fx)+a}\sqrt{c+d\sin(e+fx)}\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\sqrt{a\sin(e+fx)+a}\sqrt{c+d\sin(e+fx)}}{\sin(e+fx)}dx$$

$$\Big\downarrow \text{3428}$$

$$d\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{c+d\sin(e+fx)}}dx+c\int \frac{\csc(e+fx)\sqrt{\sin(e+fx)a+a}}{\sqrt{c+d\sin(e+fx)}}dx$$

$$\Big\downarrow \text{3042}$$

$$d\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{c+d\sin(e+fx)}}dx+c\int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}}dx$$

$$\Big\downarrow \; 3254$$

$$c\int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}}dx - \frac{2ad\int \frac{1}{\frac{a^2d\cos^2(e+fx)}{(\sin(e+fx)a+a)(c+d\sin(e+fx))}+a}d\frac{a\cos(e+fx)}{\sqrt{\sin(e+fx)a+a}\sqrt{c+d\sin(e+fx)}}}{f}$$

$$\Big\downarrow \; 218$$

$$c\int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}}dx - \frac{2\sqrt{a}\sqrt{d}\arctan\left(\frac{\sqrt{a}\sqrt{d}\cos(e+fx)}{\sqrt{a\sin(e+fx)+a}\sqrt{c+d\sin(e+fx)}}\right)}{f}$$

$$\Big\downarrow \; 3422$$

$$-\frac{2ac\int \frac{1}{1-\frac{ac\cos^2(e+fx)}{(\sin(e+fx)a+a)(c+d\sin(e+fx))}}d\frac{\cos(e+fx)}{\sqrt{\sin(e+fx)a+a}\sqrt{c+d\sin(e+fx)}}}{f} - \frac{2\sqrt{a}\sqrt{d}\arctan\left(\frac{\sqrt{a}\sqrt{d}\cos(e+fx)}{\sqrt{a\sin(e+fx)+a}\sqrt{c+d\sin(e+fx)}}\right)}{f}$$

$$\Big\downarrow \; 219$$

$$-\frac{2\sqrt{a}\sqrt{d}\arctan\left(\frac{\sqrt{a}\sqrt{d}\cos(e+fx)}{\sqrt{a\sin(e+fx)+a}\sqrt{c+d\sin(e+fx)}}\right)}{f} - \frac{2\sqrt{a}\sqrt{c}\text{arctanh}\left(\frac{\sqrt{a}\sqrt{c}\cos(e+fx)}{\sqrt{a\sin(e+fx)+a}\sqrt{c+d\sin(e+fx)}}\right)}{f}$$

input
```
Int[Csc[e + f*x]*Sqrt[a + a*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]],x]
```

output
```
(-2*Sqrt[a]*Sqrt[d]*ArcTan[(Sqrt[a]*Sqrt[d]*Cos[e + f*x])/(Sqrt[a + a*Sin[
e + f*x]]*Sqrt[c + d*Sin[e + f*x]])])/f - (2*Sqrt[a]*Sqrt[c]*ArcTanh[(Sqrt
[a]*Sqrt[c]*Cos[e + f*x])/(Sqrt[a + a*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x
]])])/f
```

3.35.3.1 Defintions of rubi rules used

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3254
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/Sqrt[(c_.) + (d_.)*sin[(e_.)
 + (f_.)*(x_)]], x_Symbol] :> Simp[-2*(b/f)   Subst[Int[1/(b + d*x^2), x],
x, b*(Cos[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]]))], x
] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0]
 && NeQ[c^2 - d^2, 0]
```

rule 3422
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(sin[(e_.) + (f_.)*(x_)]*Sqr
t[(c_) + (d_.)*sin[(e_.) + (f_.)*(x_)]]), x_Symbol] :> Simp[-2*(a/f)   Subs
t[Int[1/(1 - a*c*x^2), x], x, Cos[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c
 + d*Sin[e + f*x]])], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d,
0] && EqQ[a^2 - b^2, 0] && NeQ[b*c + a*d, 0]
```

rule 3428
```
Int[(Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(c_) + (d_.)*sin[(e_.)
 + (f_.)*(x_)]])/sin[(e_.) + (f_.)*(x_)], x_Symbol] :> Simp[d   Int[Sqrt[a
+ b*Sin[e + f*x]]/Sqrt[c + d*Sin[e + f*x]], x], x] + Simp[c   Int[Sqrt[a +
b*Sin[e + f*x]]/(Sin[e + f*x]*Sqrt[c + d*Sin[e + f*x]]), x], x] /; FreeQ[{a
, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && (NeQ[a^2 - b^2, 0] || NeQ[c^2
- d^2, 0])
```

3.35.4 Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 5065 vs. $2(99) = 198$.

Time = 0.84 (sec) , antiderivative size = 5066, normalized size of antiderivative = 41.19

output too large to display

input
```
int((a+a*sin(f*x+e))^(1/2)*(c+d*sin(f*x+e))^(1/2)/sin(f*x+e),x)
```

output
```
result too large to display
```

3.35.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 333 vs. 2(99) = 198.

Time = 1.02 (sec) , antiderivative size = 3539, normalized size of antiderivative = 28.77

$$\int \csc(e+fx)\sqrt{a+a\sin(e+fx)}\sqrt{c+d\sin(e+fx)}\,dx = \text{Too large to display}$$

input
```
integrate((a+a*sin(f*x+e))^(1/2)*(c+d*sin(f*x+e))^(1/2)/sin(f*x+e),x, algo
rithm="fricas")
```

output
```
[1/4*(sqrt(a*c)*log(((a*c^4 - 28*a*c^3*d + 70*a*c^2*d^2 - 28*a*c*d^3 + a*d
^4)*cos(f*x + e)^5 + a*c^4 + 4*a*c^3*d + 6*a*c^2*d^2 + 4*a*c*d^3 + a*d^4 -
 (31*a*c^4 - 196*a*c^3*d + 154*a*c^2*d^2 - 4*a*c*d^3 - a*d^4)*cos(f*x + e)
^4 - 2*(81*a*c^4 - 252*a*c^3*d + 150*a*c^2*d^2 - 28*a*c*d^3 + a*d^4)*cos(f
*x + e)^3 + 2*(79*a*c^4 - 100*a*c^3*d + 74*a*c^2*d^2 - 4*a*c*d^3 - a*d^4)*
cos(f*x + e)^2 - 8*((c^3 - 7*c^2*d + 7*c*d^2 - d^3)*cos(f*x + e)^4 - 2*(5*
c^3 - 14*c^2*d + 5*c*d^2)*cos(f*x + e)^3 + 51*c^3 - 59*c^2*d + 17*c*d^2 -
d^3 - 2*(18*c^3 - 33*c^2*d + 12*c*d^2 - d^3)*cos(f*x + e)^2 + 2*(13*c^3 -
14*c^2*d + 5*c*d^2)*cos(f*x + e) + ((c^3 - 7*c^2*d + 7*c*d^2 - d^3)*cos(f*
x + e)^3 - 51*c^3 + 59*c^2*d - 17*c*d^2 + d^3 + (11*c^3 - 35*c^2*d + 17*c*
d^2 - d^3)*cos(f*x + e)^2 - (25*c^3 - 31*c^2*d + 7*c*d^2 - d^3)*cos(f*x +
e))*sin(f*x + e))*sqrt(a*c)*sqrt(a*sin(f*x + e) + a)*sqrt(d*sin(f*x + e) +
 c) + (289*a*c^4 - 476*a*c^3*d + 230*a*c^2*d^2 - 28*a*c*d^3 + a*d^4)*cos(f
*x + e) + (a*c^4 + 4*a*c^3*d + 6*a*c^2*d^2 + 4*a*c*d^3 + a*d^4 + (a*c^4 -
28*a*c^3*d + 70*a*c^2*d^2 - 28*a*c*d^3 + a*d^4)*cos(f*x + e)^4 + 32*(a*c^4
 - 7*a*c^3*d + 7*a*c^2*d^2 - a*c*d^3)*cos(f*x + e)^3 - 2*(65*a*c^4 - 140*a
*c^3*d + 38*a*c^2*d^2 - 12*a*c*d^3 + a*d^4)*cos(f*x + e)^2 - 32*(9*a*c^4 -
 15*a*c^3*d + 7*a*c^2*d^2 - a*c*d^3)*cos(f*x + e))*sin(f*x + e))/(cos(f*x
+ e)^5 + cos(f*x + e)^4 - 2*cos(f*x + e)^3 - 2*cos(f*x + e)^2 + (cos(f*x +
 e)^4 - 2*cos(f*x + e)^2 + 1)*sin(f*x + e) + cos(f*x + e) + 1)) + sqrt(...
```

3.35.6 Sympy [F]

$$\int \csc(e+fx)\sqrt{a+a\sin(e+fx)}\sqrt{c+d\sin(e+fx)}\,dx$$
$$= \int \frac{\sqrt{a\left(\sin\left(e+fx\right)+1\right)}\sqrt{c+d\sin\left(e+fx\right)}}{\sin\left(e+fx\right)}\,dx$$

input
```
integrate((a+a*sin(f*x+e))**(1/2)*(c+d*sin(f*x+e))**(1/2)/sin(f*x+e),x)
```

output
```
Integral(sqrt(a*(sin(e + f*x) + 1))*sqrt(c + d*sin(e + f*x))/sin(e + f*x),
 x)
```

3.35.7 Maxima [F]

$$\int \csc(e+fx)\sqrt{a+a\sin(e+fx)}\sqrt{c+d\sin(e+fx)}\,dx$$
$$=\int \frac{\sqrt{a\sin(fx+e)+a}\sqrt{d\sin(fx+e)+c}}{\sin(fx+e)}\,dx$$

input
```
integrate((a+a*sin(f*x+e))^(1/2)*(c+d*sin(f*x+e))^(1/2)/sin(f*x+e),x, algo
rithm="maxima")
```

output
```
integrate(sqrt(a*sin(f*x + e) + a)*sqrt(d*sin(f*x + e) + c)/sin(f*x + e),
x)
```

3.35.8 Giac [F(-1)]

Timed out.

$$\int \csc(e+fx)\sqrt{a+a\sin(e+fx)}\sqrt{c+d\sin(e+fx)}\,dx=\text{Timed out}$$

input
```
integrate((a+a*sin(f*x+e))^(1/2)*(c+d*sin(f*x+e))^(1/2)/sin(f*x+e),x, algo
rithm="giac")
```

output
```
Timed out
```

3.35.9 Mupad [F(-1)]

Timed out.

$$\int \csc(e+fx)\sqrt{a+a\sin(e+fx)}\sqrt{c+d\sin(e+fx)}\,dx$$
$$=\int \frac{\sqrt{a+a\,\sin\left(e+f\,x\right)}\,\sqrt{c+d\,\sin\left(e+f\,x\right)}}{\sin\left(e+f\,x\right)}\,dx$$

input
```
int(((a + a*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))^(1/2))/sin(e + f*x),x
)
```

output
```
int(((a + a*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))^(1/2))/sin(e + f*x),
x)
```

3.36 $\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}}\,dx$

3.36.1 Optimal result

Integrand size = 35, antiderivative size = 61

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}}\,dx = -\frac{2\sqrt{a}\,\text{arctanh}\left(\frac{\sqrt{a}\sqrt{c}\cos(e+fx)}{\sqrt{a+a\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\right)}{\sqrt{c}f}$$

output
```
-2*arctanh(cos(f*x+e)*a^(1/2)*c^(1/2)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+
e))^(1/2))*a^(1/2)/f/c^(1/2)
```

3.36.2 Mathematica [C] (warning: unable to verify)

Result contains complex when optimal does not.

Time = 1.84 (sec) , antiderivative size = 367, normalized size of antiderivative = 6.02

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}}\,dx =$$

$$-\frac{\left(\log\left(-\frac{(1+i)e^{\frac{ie}{2}}\left(\sqrt{2}c\left(-1+e^{i(e+fx)}\right)+i\sqrt{2}d\left(1+e^{i(e+fx)}\right)-2i\sqrt{c}\sqrt{2ce^{i(e+fx)}-id\left(-1+e^{2i(e+fx)}\right)}\right)f}{\sqrt{c}\left(1+e^{i(e+fx)}\right)}\right)+\log\left(\frac{(1+i)e^{\frac{ie}{2}}\left(-i\sqrt{2}d(\cdots\right.}{\cdots}\right)\cdots}{\cdots}$$

input
```
Integrate[(Csc[e + f*x]*Sqrt[a + a*Sin[e + f*x]])/Sqrt[c + d*Sin[e + f*x]]
,x]
```

output
```
-(((Log[((-1 - I)*E^((I/2)*e)*(Sqrt[2]*c*(-1 + E^(I*(e + f*x))) + I*Sqrt[2
]*d*(1 + E^(I*(e + f*x))) - (2*I)*Sqrt[c]*Sqrt[2*c*E^(I*(e + f*x)) - I*d*(
-1 + E^((2*I)*(e + f*x)))])*f)/(Sqrt[c]*(1 + E^(I*(e + f*x))))] + Log[((1
+ I)*E^((I/2)*e)*((-I)*Sqrt[2]*d*(-1 + E^(I*(e + f*x))) + Sqrt[2]*c*(1 + E
^(I*(e + f*x))) + 2*Sqrt[c]*Sqrt[2*c*E^(I*(e + f*x)) - I*d*(-1 + E^((2*I)*
(e + f*x)))])*f)/(Sqrt[c]*(-1 + E^(I*(e + f*x))))])*(Cos[(e + f*x)/2] - I*
Sin[(e + f*x)/2])*Sqrt[a*(1 + Sin[e + f*x])]*Sqrt[(Cos[e + f*x] + I*Sin[e
+ f*x])*(c + d*Sin[e + f*x])])/(Sqrt[c]*f*(Cos[(e + f*x)/2] + Sin[(e + f*x
)/2])*Sqrt[c + d*Sin[e + f*x]]))
```

3.36.3 Rubi [A] (verified)

Time = 0.37 (sec) , antiderivative size = 61, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.086$, Rules used = {3042, 3422, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(e+fx)\sqrt{a\sin(e+fx)+a}}{\sqrt{c+d\sin(e+fx)}}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\sqrt{a\sin(e+fx)+a}}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}}dx$$

$$\Big\downarrow \text{3422}$$

$$-\frac{2a\int \frac{1}{1-\frac{ac\cos^2(e+fx)}{(\sin(e+fx)a+a)(c+d\sin(e+fx))}}d\frac{\cos(e+fx)}{\sqrt{\sin(e+fx)a+a}\sqrt{c+d\sin(e+fx)}}}{f}$$

$$\Big\downarrow \text{219}$$

$$-\frac{2\sqrt{a}\text{arctanh}\left(\frac{\sqrt{a}\sqrt{c}\cos(e+fx)}{\sqrt{a\sin(e+fx)+a}\sqrt{c+d\sin(e+fx)}}\right)}{\sqrt{c}f}$$

input
```
Int[(Csc[e + f*x]*Sqrt[a + a*Sin[e + f*x]])/Sqrt[c + d*Sin[e + f*x]],x]
```

output
```
(-2*Sqrt[a]*ArcTanh[(Sqrt[a]*Sqrt[c]*Cos[e + f*x])/(Sqrt[a + a*Sin[e + f*x
]]*Sqrt[c + d*Sin[e + f*x]])])/(Sqrt[c]*f)
```

3.36.3.1 Defintions of rubi rules used

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3422
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(sin[(e_.) + (f_.)*(x_)]*Sqr
t[(c_) + (d_.)*sin[(e_.) + (f_.)*(x_)]]), x_Symbol] :> Simp[-2*(a/f)   Subs
t[Int[1/(1 - a*c*x^2), x], x, Cos[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c
 + d*Sin[e + f*x]])], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d,
0] && EqQ[a^2 - b^2, 0] && NeQ[b*c + a*d, 0]
```

3.36.4 Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 202 vs. $2(49) = 98$.

Time = 2.04 (sec) , antiderivative size = 203, normalized size of antiderivative = 3.33

method	result
default	$\frac{\sqrt{a(1+\sin(fx+e))}\,\sqrt{c+d\sin(fx+e)}\,\left(\ln\left(-\frac{-\sqrt{c}\,\sqrt{2}\,\sqrt{\frac{c+d\sin(fx+e)}{1+\cos(fx+e)}}+c\cot(fx+e)-c\csc(fx+e)-d}{\sqrt{c}}\right)-\ln\left(-\frac{2\left(\sqrt{c}\,\sqrt{2}\,\sqrt{\frac{c+d\sin(fx+e)}{1+\cos(fx+e)}}\ \sin\right.}{\cos}\right.\right.}{f(\cos(fx+e)+\sin(fx+e)+1)\sqrt{\frac{c+d\sin(fx+e)}{1+\cos(fx+e)}}\,\sqrt{c}}$

input
```
int((a+a*sin(f*x+e))^(1/2)/sin(f*x+e)/(c+d*sin(f*x+e))^(1/2),x,method=_RET
URNVERBOSE)
```

output
```
1/f*(a*(1+sin(f*x+e)))^(1/2)*(c+d*sin(f*x+e))^(1/2)*(ln(-(-c^(1/2)*2^(1/2)
*((c+d*sin(f*x+e))/(1+cos(f*x+e)))^(1/2)+c*cot(f*x+e)-c*csc(f*x+e)-d)/c^(1
/2))-ln(-2*(c^(1/2)*2^(1/2)*((c+d*sin(f*x+e))/(1+cos(f*x+e)))^(1/2)*sin(f*
x+e)+c*sin(f*x+e)-cos(f*x+e)*d+d)/(cos(f*x+e)-1)))*2^(1/2)/(cos(f*x+e)+sin
(f*x+e)+1)/((c+d*sin(f*x+e))/(1+cos(f*x+e)))^(1/2)/c^(1/2)
```

3.36.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 171 vs. $2(49) = 98$.

Time = 0.46 (sec) , antiderivative size = 1044, normalized size of antiderivative = 17.11

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}}\,dx = \text{Too large to display}$$

input
```
integrate((a+a*sin(f*x+e))^(1/2)/sin(f*x+e)/(c+d*sin(f*x+e))^(1/2),x, algo
rithm="fricas")
```

output
```
[1/4*sqrt(a/c)*log(((a*c^4 - 28*a*c^3*d + 70*a*c^2*d^2 - 28*a*c*d^3 + a*d^
4)*cos(f*x + e)^5 + a*c^4 + 4*a*c^3*d + 6*a*c^2*d^2 + 4*a*c*d^3 + a*d^4 -
(31*a*c^4 - 196*a*c^3*d + 154*a*c^2*d^2 - 4*a*c*d^3 - a*d^4)*cos(f*x + e)^
4 - 2*(81*a*c^4 - 252*a*c^3*d + 150*a*c^2*d^2 - 28*a*c*d^3 + a*d^4)*cos(f*
x + e)^3 + 2*(79*a*c^4 - 100*a*c^3*d + 74*a*c^2*d^2 - 4*a*c*d^3 - a*d^4)*c
os(f*x + e)^2 - 8*((c^4 - 7*c^3*d + 7*c^2*d^2 - c*d^3)*cos(f*x + e)^4 + 51
*c^4 - 59*c^3*d + 17*c^2*d^2 - c*d^3 - 2*(5*c^4 - 14*c^3*d + 5*c^2*d^2)*co
s(f*x + e)^3 - 2*(18*c^4 - 33*c^3*d + 12*c^2*d^2 - c*d^3)*cos(f*x + e)^2 +
 2*(13*c^4 - 14*c^3*d + 5*c^2*d^2)*cos(f*x + e) - (51*c^4 - 59*c^3*d + 17*
c^2*d^2 - c*d^3 - (c^4 - 7*c^3*d + 7*c^2*d^2 - c*d^3)*cos(f*x + e)^3 - (11
*c^4 - 35*c^3*d + 17*c^2*d^2 - c*d^3)*cos(f*x + e)^2 + (25*c^4 - 31*c^3*d
+ 7*c^2*d^2 - c*d^3)*cos(f*x + e))*sin(f*x + e))*sqrt(a*sin(f*x + e) + a)*
sqrt(d*sin(f*x + e) + c)*sqrt(a/c) + (289*a*c^4 - 476*a*c^3*d + 230*a*c^2*
d^2 - 28*a*c*d^3 + a*d^4)*cos(f*x + e) + (a*c^4 + 4*a*c^3*d + 6*a*c^2*d^2
+ 4*a*c*d^3 + a*d^4 + (a*c^4 - 28*a*c^3*d + 70*a*c^2*d^2 - 28*a*c*d^3 + a*
d^4)*cos(f*x + e)^4 + 32*(a*c^4 - 7*a*c^3*d + 7*a*c^2*d^2 - a*c*d^3)*cos(f
*x + e)^3 - 2*(65*a*c^4 - 140*a*c^3*d + 38*a*c^2*d^2 - 12*a*c*d^3 + a*d^4)
*cos(f*x + e)^2 - 32*(9*a*c^4 - 15*a*c^3*d + 7*a*c^2*d^2 - a*c*d^3)*cos(f*
x + e))*sin(f*x + e))/(cos(f*x + e)^5 + cos(f*x + e)^4 - 2*cos(f*x + e)^3
- 2*cos(f*x + e)^2 + (cos(f*x + e)^4 - 2*cos(f*x + e)^2 + 1)*sin(f*x + ...
```

3.36.6 Sympy [F]

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}}\,dx = \int \frac{\sqrt{a\left(\sin\left(e+fx\right)+1\right)}}{\sqrt{c+d\sin\left(e+fx\right)}\sin\left(e+fx\right)}\,dx$$

input
```
integrate((a+a*sin(f*x+e))**(1/2)/sin(f*x+e)/(c+d*sin(f*x+e))**(1/2),x)
```

output
```
Integral(sqrt(a*(sin(e + f*x) + 1))/(sqrt(c + d*sin(e + f*x))*sin(e + f*x)
), x)
```

3.36.7 Maxima [F]

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}}\,dx=\int \frac{\sqrt{a\sin\left(fx+e\right)+a}}{\sqrt{d\sin\left(fx+e\right)+c}\sin\left(fx+e\right)}\,dx$$

input
```
integrate((a+a*sin(f*x+e))^(1/2)/sin(f*x+e)/(c+d*sin(f*x+e))^(1/2),x, algo
rithm="maxima")
```

output
```
integrate(sqrt(a*sin(f*x + e) + a)/(sqrt(d*sin(f*x + e) + c)*sin(f*x + e))
, x)
```

3.36.8 Giac [F]

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}}\,dx=\int \frac{\sqrt{a\sin\left(fx+e\right)+a}}{\sqrt{d\sin\left(fx+e\right)+c}\sin\left(fx+e\right)}\,dx$$

input
```
integrate((a+a*sin(f*x+e))^(1/2)/sin(f*x+e)/(c+d*sin(f*x+e))^(1/2),x, algo
rithm="giac")
```

output
```
sage0*x
```

3.36.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}}\,dx=\int \frac{\sqrt{a+a\,\sin\left(e+f\,x\right)}}{\sin\left(e+f\,x\right)\,\sqrt{c+d\,\sin\left(e+f\,x\right)}}\,dx$$

input
```
int((a + a*sin(e + f*x))^(1/2)/(sin(e + f*x)*(c + d*sin(e + f*x))^(1/2)),x
)
```

output
```
int((a + a*sin(e + f*x))^(1/2)/(sin(e + f*x)*(c + d*sin(e + f*x))^(1/2)),
x)
```

3.36. $\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}}\,dx$

3.37 $\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}}\,dx$

3.37.1 Optimal result

Integrand size = 35, antiderivative size = 140

$$\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}}\,dx = -\frac{2\sqrt{c}\operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{c}\cos(e+fx)}{\sqrt{a+a\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\right)}{\sqrt{a}f} + \frac{\sqrt{2}\sqrt{c-d}\operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{c-d}\cos(e+fx)}{\sqrt{2}\sqrt{a+a\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\right)}{\sqrt{a}f}$$

output
```
-2*arctanh(cos(f*x+e)*a^(1/2)*c^(1/2)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+
e))^(1/2))*c^(1/2)/f/a^(1/2)+arctanh(1/2*cos(f*x+e)*a^(1/2)*(c-d)^(1/2)*2^
(1/2)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2))*2^(1/2)*(c-d)^(1/2)/f
/a^(1/2)
```

3.37.2 Mathematica [B] (warning: unable to verify)

Leaf count is larger than twice the leaf count of optimal. 665 vs. $2(140) = 280$.

Time = 10.59 (sec) , antiderivative size = 665, normalized size of antiderivative = 4.75

$$\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}}\,dx$$
$$= \frac{\csc(e+fx)\left(\sqrt{c}\log\left(\tan\left(\frac{1}{2}(e+fx)\right)\right)-\sqrt{2}\sqrt{c-d}\log\left(1+\tan\left(\frac{1}{2}(e+fx)\right)\right)+\sqrt{c}\log\left(d+\sqrt{2}\sqrt{c}\sqrt{\frac{1}{1+c}}\right.\right.}{f\sqrt{a(1+\sin(e+fx))}\left(\sqrt{c}\csc(e+\right.}$$

input
```
Integrate[(Csc[e + f*x]*Sqrt[c + d*Sin[e + f*x]])/Sqrt[a + a*Sin[e + f*x]]
,x]
```

output
```
(Csc[e + f*x]*(Sqrt[c]*Log[Tan[(e + f*x)/2]] - Sqrt[2]*Sqrt[c - d]*Log[1 +
 Tan[(e + f*x)/2]] + Sqrt[c]*Log[d + Sqrt[2]*Sqrt[c]*Sqrt[(1 + Cos[e + f*x
])^(-1)]*Sqrt[c + d*Sin[e + f*x]] + c*Tan[(e + f*x)/2]] - Sqrt[c]*Log[c +
Sqrt[2]*Sqrt[c]*Sqrt[(1 + Cos[e + f*x])^(-1)]*Sqrt[c + d*Sin[e + f*x]] + d
*Tan[(e + f*x)/2]] + Sqrt[2]*Sqrt[c - d]*Log[c - d + 2*Sqrt[c - d]*Sqrt[(1
 + Cos[e + f*x])^(-1)]*Sqrt[c + d*Sin[e + f*x]] + (-c + d)*Tan[(e + f*x)/2
]])*Sqrt[c + d*Sin[e + f*x]])/(f*Sqrt[a*(1 + Sin[e + f*x])]*(Sqrt[c]*Csc[e
 + f*x] + (c*Sqrt[Sec[(e + f*x)/2]^2])/(2*Sqrt[c + d*Sin[e + f*x]]) - (Sqr
t[c - d]*Sec[(e + f*x)/2]^2)/(Sqrt[2]*(1 + Tan[(e + f*x)/2])) - (Sqrt[c]*(
d*Sec[(e + f*x)/2]^2 + (Sqrt[2]*Sqrt[c]*((1 + Cos[e + f*x])^(-1))^(3/2)*(d
 + d*Cos[e + f*x] + c*Sin[e + f*x]))/Sqrt[c + d*Sin[e + f*x]]))/(2*(c + Sq
rt[2]*Sqrt[c]*Sqrt[(1 + Cos[e + f*x])^(-1)]*Sqrt[c + d*Sin[e + f*x]] + d*T
an[(e + f*x)/2])) + (Sqrt[2]*Sqrt[c - d]*(-1/2*((c - d)*Sec[(e + f*x)/2]^2
) + (Sqrt[c - d]*((1 + Cos[e + f*x])^(-1))^(3/2)*(d + d*Cos[e + f*x] + c*S
in[e + f*x]))/Sqrt[c + d*Sin[e + f*x]]))/(c - d + 2*Sqrt[c - d]*Sqrt[(1 +
Cos[e + f*x])^(-1)]*Sqrt[c + d*Sin[e + f*x]] + (-c + d)*Tan[(e + f*x)/2]))
)
```

3.37.3 Rubi [A] (verified)

Time = 0.92 (sec) , antiderivative size = 140, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.200, Rules used = {3042, 3423, 3042, 3261, 221, 3422, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{\sqrt{a\sin(e+fx)+a}}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\sqrt{c+d\sin(e+fx)}}{\sin(e+fx)\sqrt{a\sin(e+fx)+a}}dx$$

$$\Big\downarrow \text{3423}$$

$$\frac{c\int \frac{\csc(e+fx)\sqrt{\sin(e+fx)a+a}}{\sqrt{c+d\sin(e+fx)}}dx}{a}-(c-d)\int \frac{1}{\sqrt{\sin(e+fx)a+a}\sqrt{c+d\sin(e+fx)}}dx$$

$$\Big\downarrow \text{3042}$$

$$\frac{c\int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}}dx}{a}-(c-d)\int \frac{1}{\sqrt{\sin(e+fx)a+a}\sqrt{c+d\sin(e+fx)}}dx$$

$$\Big\downarrow \text{3261}$$

$$\frac{2a(c-d)\int \frac{1}{2a^2-\frac{a^3(c-d)\cos^2(e+fx)}{(\sin(e+fx)a+a)(c+d\sin(e+fx))}}d\frac{a\cos(e+fx)}{\sqrt{\sin(e+fx)a+a}\sqrt{c+d\sin(e+fx)}}}{f}+\frac{c\int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}}dx}{a}$$

$$\Big\downarrow \text{221}$$

$$\frac{c\int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}}dx}{a}+\frac{\sqrt{2}\sqrt{c-d}\,\text{arctanh}\left(\frac{\sqrt{a}\sqrt{c-d}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}\sqrt{c+d\sin(e+fx)}}\right)}{\sqrt{a}f}$$

$$\Big\downarrow \text{3422}$$

$$\frac{\sqrt{2}\sqrt{c-d}\,\text{arctanh}\left(\frac{\sqrt{a}\sqrt{c-d}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}\sqrt{c+d\sin(e+fx)}}\right)}{\sqrt{a}f}-\frac{2c\int \frac{1}{1-\frac{ac\cos^2(e+fx)}{(\sin(e+fx)a+a)(c+d\sin(e+fx))}}d\frac{\cos(e+fx)}{\sqrt{\sin(e+fx)a+a}\sqrt{c+d\sin(e+fx)}}}{f}$$

$$\Big\downarrow \text{219}$$

$$\frac{\sqrt{2}\sqrt{c-d}\,\text{arctanh}\left(\frac{\sqrt{a}\sqrt{c-d}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}\sqrt{c+d\sin(e+fx)}}\right)}{\sqrt{a}f}-\frac{2\sqrt{c}\,\text{arctanh}\left(\frac{\sqrt{a}\sqrt{c}\cos(e+fx)}{\sqrt{a\sin(e+fx)+a}\sqrt{c+d\sin(e+fx)}}\right)}{\sqrt{a}f}$$

input
```
Int[(Csc[e + f*x]*Sqrt[c + d*Sin[e + f*x]])/Sqrt[a + a*Sin[e + f*x]],x]
```

output
```
(-2*Sqrt[c]*ArcTanh[(Sqrt[a]*Sqrt[c]*Cos[e + f*x])/(Sqrt[a + a*Sin[e + f*x
]]*Sqrt[c + d*Sin[e + f*x]])])/(Sqrt[a]*f) + (Sqrt[2]*Sqrt[c - d]*ArcTanh[
(Sqrt[a]*Sqrt[c - d]*Cos[e + f*x])/(Sqrt[2]*Sqrt[a + a*Sin[e + f*x]]*Sqrt[
c + d*Sin[e + f*x]])])/(Sqrt[a]*f)
```

3.37.3.1 Defintions of rubi rules used

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 221
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[-a/b, 2]/a)*ArcTanh[x
/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3261
```
Int[1/(Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(c_.) + (d_.)*sin[(e
_.) + (f_.)*(x_)]]), x_Symbol] :> Simp[-2*(a/f)   Subst[Int[1/(2*b^2 - (a*c
 - b*d)*x^2), x], x, b*(Cos[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*S
in[e + f*x]]))], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] &&
 EqQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]
```

rule 3422
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(sin[(e_.) + (f_.)*(x_)]*Sqr
t[(c_) + (d_.)*sin[(e_.) + (f_.)*(x_)]]), x_Symbol] :> Simp[-2*(a/f)   Subs
t[Int[1/(1 - a*c*x^2), x], x, Cos[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c
 + d*Sin[e + f*x]])], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d,
0] && EqQ[a^2 - b^2, 0] && NeQ[b*c + a*d, 0]
```

rule 3423
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(sin[(e_.) + (f_.)*(x_)]*Sqr
t[(c_) + (d_.)*sin[(e_.) + (f_.)*(x_)]]), x_Symbol] :> Simp[(b*c - a*d)/c
 Int[1/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]]), x], x] + Simp[a
/c   Int[Sqrt[c + d*Sin[e + f*x]]/(Sin[e + f*x]*Sqrt[a + b*Sin[e + f*x]]),
x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^
2, 0] && EqQ[c^2 - d^2, 0]
```

3.37.4 Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 325 vs. $2(113) = 226$.

Time = 1.80 (sec) , antiderivative size = 326, normalized size of antiderivative = 2.33

method	result
default	$\frac{\sqrt{c+d\sin(fx+e)}\,\sqrt{2}\left(\sqrt{2c-2d}\,\ln\left(\frac{2\sqrt{2c-2d}\,\sqrt{2}\,\sqrt{\frac{c+d\sin(fx+e)}{1+\cos(fx+e)}}\,\sin(fx+e)+2c\sin(fx+e)-2d\sin(fx+e)+2c\cos(fx+e)-2\cos(fx+e)d-2c+2}{-\cos(fx+e)+1+\sin(fx+e)}\right.\right.}{2f(1+\cos(}$

input
```
int((c+d*sin(f*x+e))^(1/2)/sin(f*x+e)/(a+a*sin(f*x+e))^(1/2),x,method=_RET
URNVERBOSE)
```

output
```
1/2/f*(c+d*sin(f*x+e))^(1/2)*2^(1/2)*((2*c-2*d)^(1/2)*ln(2*((2*c-2*d)^(1/2
)*2^(1/2)*((c+d*sin(f*x+e))/(1+cos(f*x+e)))^(1/2)*sin(f*x+e)+c*sin(f*x+e)-
d*sin(f*x+e)+c*cos(f*x+e)-cos(f*x+e)*d-c+d)/(-cos(f*x+e)+1+sin(f*x+e)))*c^
(1/2)+ln((c^(1/2)*2^(1/2)*((c+d*sin(f*x+e))/(1+cos(f*x+e)))^(1/2)+c*csc(f*
x+e)-c*cot(f*x+e)+d)/c^(1/2))*c-c*ln(-2*(c^(1/2)*2^(1/2)*((c+d*sin(f*x+e))
/(1+cos(f*x+e)))^(1/2)*sin(f*x+e)+c*sin(f*x+e)-cos(f*x+e)*d+d)/(cos(f*x+e)
-1)))*(cos(f*x+e)+sin(f*x+e)+1)/(1+cos(f*x+e))/(a*(1+sin(f*x+e)))^(1/2)/((
c+d*sin(f*x+e))/(1+cos(f*x+e)))^(1/2)/c^(1/2)
```

3.37.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 288 vs. $2(113) = 226$.

Time = 0.61 (sec) , antiderivative size = 2791, normalized size of antiderivative = 19.94

$$\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}}\,dx = \text{Too large to display}$$

input
```
integrate((c+d*sin(f*x+e))^(1/2)/sin(f*x+e)/(a+a*sin(f*x+e))^(1/2),x, algo
rithm="fricas")
```

output
```
[1/4*(sqrt(2)*sqrt((c - d)/a)*log(((c^2 - 14*c*d + 17*d^2)*cos(f*x + e)^3
+ 4*sqrt(2)*((c - 3*d)*cos(f*x + e)^2 - (3*c - d)*cos(f*x + e) + ((c - 3*d
)*cos(f*x + e) + 4*c - 4*d)*sin(f*x + e) - 4*c + 4*d)*sqrt(a*sin(f*x + e)
+ a)*sqrt(d*sin(f*x + e) + c)*sqrt((c - d)/a) - (13*c^2 - 22*c*d - 3*d^2)*
cos(f*x + e)^2 - 4*c^2 - 8*c*d - 4*d^2 - 2*(9*c^2 - 14*c*d + 9*d^2)*cos(f*
x + e) + ((c^2 - 14*c*d + 17*d^2)*cos(f*x + e)^2 - 4*c^2 - 8*c*d - 4*d^2 +
 2*(7*c^2 - 18*c*d + 7*d^2)*cos(f*x + e))*sin(f*x + e))/(cos(f*x + e)^3 +
3*cos(f*x + e)^2 + (cos(f*x + e)^2 - 2*cos(f*x + e) - 4)*sin(f*x + e) - 2*
cos(f*x + e) - 4)) + sqrt(c/a)*log(((c^4 - 28*c^3*d + 70*c^2*d^2 - 28*c*d^
3 + d^4)*cos(f*x + e)^5 - (31*c^4 - 196*c^3*d + 154*c^2*d^2 - 4*c*d^3 - d^
4)*cos(f*x + e)^4 + c^4 + 4*c^3*d + 6*c^2*d^2 + 4*c*d^3 + d^4 - 2*(81*c^4
- 252*c^3*d + 150*c^2*d^2 - 28*c*d^3 + d^4)*cos(f*x + e)^3 + 2*(79*c^4 - 1
00*c^3*d + 74*c^2*d^2 - 4*c*d^3 - d^4)*cos(f*x + e)^2 - 8*((c^3 - 7*c^2*d
+ 7*c*d^2 - d^3)*cos(f*x + e)^4 - 2*(5*c^3 - 14*c^2*d + 5*c*d^2)*cos(f*x +
 e)^3 + 51*c^3 - 59*c^2*d + 17*c*d^2 - d^3 - 2*(18*c^3 - 33*c^2*d + 12*c*d
^2 - d^3)*cos(f*x + e)^2 + 2*(13*c^3 - 14*c^2*d + 5*c*d^2)*cos(f*x + e) +
((c^3 - 7*c^2*d + 7*c*d^2 - d^3)*cos(f*x + e)^3 - 51*c^3 + 59*c^2*d - 17*c
*d^2 + d^3 + (11*c^3 - 35*c^2*d + 17*c*d^2 - d^3)*cos(f*x + e)^2 - (25*c^3
 - 31*c^2*d + 7*c*d^2 - d^3)*cos(f*x + e))*sin(f*x + e))*sqrt(a*sin(f*x +
e) + a)*sqrt(d*sin(f*x + e) + c)*sqrt(c/a) + (289*c^4 - 476*c^3*d + 230...
```

3.37.6 Sympy [F]

$$\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}}\,dx = \int \frac{\sqrt{c+d\sin(e+fx)}}{\sqrt{a(\sin(e+fx)+1)}\sin(e+fx)}\,dx$$

input
```
integrate((c+d*sin(f*x+e))**(1/2)/sin(f*x+e)/(a+a*sin(f*x+e))**(1/2),x)
```

output
```
Integral(sqrt(c + d*sin(e + f*x))/(sqrt(a*(sin(e + f*x) + 1))*sin(e + f*x)
), x)
```

3.37.7 Maxima [F]

$$\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}}\,dx = \int \frac{\sqrt{d\sin(fx+e)+c}}{\sqrt{a\sin(fx+e)+a}\sin(fx+e)}\,dx$$

input
```
integrate((c+d*sin(f*x+e))^(1/2)/sin(f*x+e)/(a+a*sin(f*x+e))^(1/2),x, algo
rithm="maxima")
```

output
```
integrate(sqrt(d*sin(f*x + e) + c)/(sqrt(a*sin(f*x + e) + a)*sin(f*x + e))
, x)
```

3.37.8 Giac [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}}\,dx = \text{Timed out}$$

input
```
integrate((c+d*sin(f*x+e))^(1/2)/sin(f*x+e)/(a+a*sin(f*x+e))^(1/2),x, algo
rithm="giac")
```

output
```
Timed out
```

3.37.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}}\,dx = \int \frac{\sqrt{c+d\,\sin(e+f\,x)}}{\sin(e+f\,x)\,\sqrt{a+a\,\sin(e+f\,x)}}\,dx$$

input
```
int((c + d*sin(e + f*x))^(1/2)/(sin(e + f*x)*(a + a*sin(e + f*x))^(1/2)),x
)
```

output
```
int((c + d*sin(e + f*x))^(1/2)/(sin(e + f*x)*(a + a*sin(e + f*x))^(1/2)),
x)
```

3.38 $\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\,dx$

3.38.1 Optimal result

Integrand size = 35, antiderivative size = 140

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\,dx = -\frac{2\operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{c}\cos(e+fx)}{\sqrt{a+a\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\right)}{\sqrt{a}\sqrt{c}f} + \frac{\sqrt{2}\operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{c-d}\cos(e+fx)}{\sqrt{2}\sqrt{a+a\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\right)}{\sqrt{a}\sqrt{c-d}f}$$

output
```
-2*arctanh(cos(f*x+e)*a^(1/2)*c^(1/2)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+
e))^(1/2))/f/a^(1/2)/c^(1/2)+arctanh(1/2*cos(f*x+e)*a^(1/2)*(c-d)^(1/2)*2^
(1/2)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2))*2^(1/2)/f/a^(1/2)/(c-
d)^(1/2)
```

3.38.2 Mathematica [B] (warning: unable to verify)

Leaf count is larger than twice the leaf count of optimal. 885 vs. $2(140) = 280$.

Time = 14.91 (sec) , antiderivative size = 885, normalized size of antiderivative = 6.32

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\,dx$$

$$= \frac{\csc(e+fx)\left(\frac{\log\left(\tan\left(\frac{1}{2}(e+fx)\right)\right)}{\sqrt{c}} - \frac{\sqrt{2}\log\left(1+\tan\left(\frac{1}{2}(e+f\cdots\right.\right.}{\sqrt{c-d}}\cdots\right.}{f\sqrt{a(1+\sin(e+fx))}\sqrt{c+d\sin(e+fx)}\left(\frac{\csc\left(\frac{1}{2}(e+fx)\right)\sec\left(\frac{1}{2}(e+fx)\right)}{2\sqrt{c}} - \frac{\sec^2\left(\frac{1}{2}(e+fx)\right)}{\sqrt{2}\sqrt{c-d}\left(1+\tan\left(\frac{1}{2}(e+fx)\right)\right)} + \frac{\frac{1}{2}c\sec^2\left(\frac{1}{2}(e+f\cdots\right.}{\cdots}\right.}$$

input
```
Integrate[Csc[e + f*x]/(Sqrt[a + a*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]])
,x]
```

output
```
(Csc[e + f*x]*(Log[Tan[(e + f*x)/2]]/Sqrt[c] - (Sqrt[2]*Log[1 + Tan[(e + f
*x)/2]])/Sqrt[c - d] + Log[d + Sqrt[2]*Sqrt[c]*Sqrt[(1 + Cos[e + f*x])^(-1
)]*Sqrt[c + d*Sin[e + f*x]] + c*Tan[(e + f*x)/2]]/Sqrt[c] - Log[c + Sqrt[2
]*Sqrt[c]*Sqrt[(1 + Cos[e + f*x])^(-1)]*Sqrt[c + d*Sin[e + f*x]] + d*Tan[(
e + f*x)/2]]/Sqrt[c] + (Sqrt[2]*Log[c - d + 2*Sqrt[c - d]*Sqrt[(1 + Cos[e
+ f*x])^(-1)]*Sqrt[c + d*Sin[e + f*x]] + (-c + d)*Tan[(e + f*x)/2]])/Sqrt[
c - d]))/(f*Sqrt[a*(1 + Sin[e + f*x])]*Sqrt[c + d*Sin[e + f*x]]*((Csc[(e +
 f*x)/2]*Sec[(e + f*x)/2])/(2*Sqrt[c]) - Sec[(e + f*x)/2]^2/(Sqrt[2]*Sqrt[
c - d]*(1 + Tan[(e + f*x)/2])) + ((c*Sec[(e + f*x)/2]^2)/2 + (Sqrt[c]*d*Co
s[e + f*x]*Sqrt[(1 + Cos[e + f*x])^(-1)])/(Sqrt[2]*Sqrt[c + d*Sin[e + f*x]
]) + (Sqrt[c]*((1 + Cos[e + f*x])^(-1))^(3/2)*Sin[e + f*x]*Sqrt[c + d*Sin[
e + f*x]])/Sqrt[2])/(Sqrt[c]*(d + Sqrt[2]*Sqrt[c]*Sqrt[(1 + Cos[e + f*x])^
(-1)]*Sqrt[c + d*Sin[e + f*x]] + c*Tan[(e + f*x)/2])) - ((d*Sec[(e + f*x)/
2]^2)/2 + (Sqrt[c]*d*Cos[e + f*x]*Sqrt[(1 + Cos[e + f*x])^(-1)])/(Sqrt[2]*
Sqrt[c + d*Sin[e + f*x]]) + (Sqrt[c]*((1 + Cos[e + f*x])^(-1))^(3/2)*Sin[e
 + f*x]*Sqrt[c + d*Sin[e + f*x]])/Sqrt[2])/(Sqrt[c]*(c + Sqrt[2]*Sqrt[c]*S
qrt[(1 + Cos[e + f*x])^(-1)]*Sqrt[c + d*Sin[e + f*x]] + d*Tan[(e + f*x)/2]
)) + (Sqrt[2]*(((-c + d)*Sec[(e + f*x)/2]^2)/2 + (Sqrt[c - d]*d*Cos[e + f*
x]*Sqrt[(1 + Cos[e + f*x])^(-1)])/Sqrt[c + d*Sin[e + f*x]] + Sqrt[c - d]*(
(1 + Cos[e + f*x])^(-1))^(3/2)*Sin[e + f*x]*Sqrt[c + d*Sin[e + f*x]]))/...
```

3.38.3 Rubi [A] (verified)

Time = 0.89 (sec) , antiderivative size = 140, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.200, Rules used = {3042, 3426, 3042, 3261, 221, 3422, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(e+fx)}{\sqrt{a\sin(e+fx)+a}\sqrt{c+d\sin(e+fx)}}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{1}{\sin(e+fx)\sqrt{a\sin(e+fx)+a}\sqrt{c+d\sin(e+fx)}}dx$$

$$\Big\downarrow \text{3426}$$

$$\frac{\int \frac{\csc(e+fx)\sqrt{\sin(e+fx)a+a}}{\sqrt{c+d\sin(e+fx)}}dx}{a}-\int \frac{1}{\sqrt{\sin(e+fx)a+a}\sqrt{c+d\sin(e+fx)}}dx$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}}dx}{a}-\int \frac{1}{\sqrt{\sin(e+fx)a+a}\sqrt{c+d\sin(e+fx)}}dx$$

$$\Big\downarrow \text{3261}$$

$$\frac{2a\int \frac{1}{2a^2-\frac{a^3(c-d)\cos^2(e+fx)}{(\sin(e+fx)a+a)(c+d\sin(e+fx))}}d\frac{a\cos(e+fx)}{\sqrt{\sin(e+fx)a+a}\sqrt{c+d\sin(e+fx)}}}{f}+\frac{\int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}}dx}{a}$$

$$\Big\downarrow \text{221}$$

$$\frac{\int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}}dx}{a}+\frac{\sqrt{2}\text{arctanh}\left(\frac{\sqrt{a}\sqrt{c-d}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}\sqrt{c+d\sin(e+fx)}}\right)}{\sqrt{a}f\sqrt{c-d}}$$

$$\Big\downarrow \text{3422}$$

$$\frac{\sqrt{2}\text{arctanh}\left(\frac{\sqrt{a}\sqrt{c-d}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}\sqrt{c+d\sin(e+fx)}}\right)}{\sqrt{a}f\sqrt{c-d}}-\frac{2\int \frac{1}{1-\frac{ac\cos^2(e+fx)}{(\sin(e+fx)a+a)(c+d\sin(e+fx))}}d\frac{\cos(e+fx)}{\sqrt{\sin(e+fx)a+a}\sqrt{c+d\sin(e+fx)}}}{f}$$

$$\Big\downarrow \text{219}$$

$$\frac{\sqrt{2}\text{arctanh}\left(\frac{\sqrt{a}\sqrt{c-d}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}\sqrt{c+d\sin(e+fx)}}\right)}{\sqrt{a}f\sqrt{c-d}} - \frac{2\text{arctanh}\left(\frac{\sqrt{a}\sqrt{c}\cos(e+fx)}{\sqrt{a\sin(e+fx)+a}\sqrt{c+d\sin(e+fx)}}\right)}{\sqrt{a}\sqrt{c}f}$$

input
```
Int[Csc[e + f*x]/(Sqrt[a + a*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]]),x]
```

output
```
(-2*ArcTanh[(Sqrt[a]*Sqrt[c]*Cos[e + f*x])/(Sqrt[a + a*Sin[e + f*x]]*Sqrt[
c + d*Sin[e + f*x]])])/(Sqrt[a]*Sqrt[c]*f) + (Sqrt[2]*ArcTanh[(Sqrt[a]*Sqr
t[c - d]*Cos[e + f*x])/(Sqrt[2]*Sqrt[a + a*Sin[e + f*x]]*Sqrt[c + d*Sin[e
+ f*x]])])/(Sqrt[a]*Sqrt[c - d]*f)
```

3.38.3.1 Defintions of rubi rules used

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 221
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[-a/b, 2]/a)*ArcTanh[x
/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3261
```
Int[1/(Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(c_.) + (d_.)*sin[(e
_.) + (f_.)*(x_)]]), x_Symbol] :> Simp[-2*(a/f)   Subst[Int[1/(2*b^2 - (a*c
 - b*d)*x^2), x], x, b*(Cos[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*S
in[e + f*x]]))], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] &&
 EqQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]
```

rule 3422
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(sin[(e_.) + (f_.)*(x_)]*Sqr
t[(c_) + (d_.)*sin[(e_.) + (f_.)*(x_)]]), x_Symbol] :> Simp[-2*(a/f)   Subs
t[Int[1/(1 - a*c*x^2), x], x, Cos[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c
 + d*Sin[e + f*x]])], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d,
0] && EqQ[a^2 - b^2, 0] && NeQ[b*c + a*d, 0]
```

rule 3426
```
Int[1/(sin[(e_.) + (f_.)*(x_)]*Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]*S
qrt[(c_) + (d_.)*sin[(e_.) + (f_.)*(x_)]]), x_Symbol] :> Simp[-b/a   Int[1/
(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]]), x], x] + Simp[1/a   In
t[Sqrt[a + b*Sin[e + f*x]]/(Sin[e + f*x]*Sqrt[c + d*Sin[e + f*x]]), x], x]
/; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && (NeQ[a^2 - b^2, 0]
|| NeQ[c^2 - d^2, 0])
```

3.38.4 Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 346 vs. $2(113) = 226$.

Time = 1.65 (sec) , antiderivative size = 347, normalized size of antiderivative = 2.48

method	result
default	$\frac{\sqrt{c+d\sin(fx+e)}\,\sqrt{2}\left(\ln\left(-\frac{-\sqrt{c}\,\sqrt{2}\,\sqrt{\frac{c+d\sin(fx+e)}{1+\cos(fx+e)}}+c\cot(fx+e)-c\csc(fx+e)-d}{\sqrt{c}}\right)\sqrt{2c-2d}-\ln\left(-\frac{2\left(\sqrt{c}\,\sqrt{2}\,\sqrt{\frac{c+d\sin(fx+e)}{1+\cos(fx+e)}}\,\sin(fx+e)\right.}{\cos(fx+e)}\right.}{2f(1+\cos}$

input
```
int(1/sin(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x,method=_R
ETURNVERBOSE)
```

output
```
1/2/f*(c+d*sin(f*x+e))^(1/2)*2^(1/2)*(ln(-(-c^(1/2)*2^(1/2)*((c+d*sin(f*x+
e))/(1+cos(f*x+e)))^(1/2)+c*cot(f*x+e)-c*csc(f*x+e)-d)/c^(1/2))*(2*c-2*d)^
(1/2)-ln(-2*(c^(1/2)*2^(1/2)*((c+d*sin(f*x+e))/(1+cos(f*x+e)))^(1/2)*sin(f
*x+e)+c*sin(f*x+e)-cos(f*x+e)*d+d)/(cos(f*x+e)-1))*(2*c-2*d)^(1/2)+2*ln(2*
((2*c-2*d)^(1/2)*2^(1/2)*((c+d*sin(f*x+e))/(1+cos(f*x+e)))^(1/2)*sin(f*x+e
)+c*sin(f*x+e)-d*sin(f*x+e)+c*cos(f*x+e)-cos(f*x+e)*d-c+d)/(-cos(f*x+e)+1+
sin(f*x+e))*c^(1/2))*(cos(f*x+e)+sin(f*x+e)+1)/(1+cos(f*x+e))/(a*(1+sin(f
*x+e)))^(1/2)/((c+d*sin(f*x+e))/(1+cos(f*x+e)))^(1/2)/c^(1/2)/(2*c-2*d)^(1
/2)
```

3.38.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 290 vs. 2(113) = 226.

Time = 0.71 (sec) , antiderivative size = 3005, normalized size of antiderivative = 21.46

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\,dx = \text{Too large to display}$$

input
```
integrate(1/sin(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x, al
gorithm="fricas")
```

output
```
[1/4*(sqrt(2)*a*c*log(((c^2 - 14*c*d + 17*d^2)*cos(f*x + e)^3 - (13*c^2 -
22*c*d - 3*d^2)*cos(f*x + e)^2 + 4*sqrt(2)*((c^2 - 4*c*d + 3*d^2)*cos(f*x
+ e)^2 - 4*c^2 + 8*c*d - 4*d^2 - (3*c^2 - 4*c*d + d^2)*cos(f*x + e) + (4*c
^2 - 8*c*d + 4*d^2 + (c^2 - 4*c*d + 3*d^2)*cos(f*x + e))*sin(f*x + e))*sqr
t(a*sin(f*x + e) + a)*sqrt(d*sin(f*x + e) + c)/sqrt(a*c - a*d) - 4*c^2 - 8
*c*d - 4*d^2 - 2*(9*c^2 - 14*c*d + 9*d^2)*cos(f*x + e) + ((c^2 - 14*c*d +
17*d^2)*cos(f*x + e)^2 - 4*c^2 - 8*c*d - 4*d^2 + 2*(7*c^2 - 18*c*d + 7*d^2
)*cos(f*x + e))*sin(f*x + e))/(cos(f*x + e)^3 + 3*cos(f*x + e)^2 + (cos(f*
x + e)^2 - 2*cos(f*x + e) - 4)*sin(f*x + e) - 2*cos(f*x + e) - 4))/sqrt(a*
c - a*d) + sqrt(a*c)*log(((a*c^4 - 28*a*c^3*d + 70*a*c^2*d^2 - 28*a*c*d^3
+ a*d^4)*cos(f*x + e)^5 + a*c^4 + 4*a*c^3*d + 6*a*c^2*d^2 + 4*a*c*d^3 + a*
d^4 - (31*a*c^4 - 196*a*c^3*d + 154*a*c^2*d^2 - 4*a*c*d^3 - a*d^4)*cos(f*x
 + e)^4 - 2*(81*a*c^4 - 252*a*c^3*d + 150*a*c^2*d^2 - 28*a*c*d^3 + a*d^4)*
cos(f*x + e)^3 + 2*(79*a*c^4 - 100*a*c^3*d + 74*a*c^2*d^2 - 4*a*c*d^3 - a*
d^4)*cos(f*x + e)^2 - 8*((c^3 - 7*c^2*d + 7*c*d^2 - d^3)*cos(f*x + e)^4 -
2*(5*c^3 - 14*c^2*d + 5*c*d^2)*cos(f*x + e)^3 + 51*c^3 - 59*c^2*d + 17*c*d
^2 - d^3 - 2*(18*c^3 - 33*c^2*d + 12*c*d^2 - d^3)*cos(f*x + e)^2 + 2*(13*c
^3 - 14*c^2*d + 5*c*d^2)*cos(f*x + e) + ((c^3 - 7*c^2*d + 7*c*d^2 - d^3)*c
os(f*x + e)^3 - 51*c^3 + 59*c^2*d - 17*c*d^2 + d^3 + (11*c^3 - 35*c^2*d +
17*c*d^2 - d^3)*cos(f*x + e)^2 - (25*c^3 - 31*c^2*d + 7*c*d^2 - d^3)*co...
```

3.38.6 Sympy [F]

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\,dx = \int \frac{1}{\sqrt{a\left(\sin\left(e+fx\right)+1\right)}\sqrt{c+d\sin\left(e+fx\right)}\sin\left(e+fx\right)}\,dx$$

input
```
integrate(1/sin(f*x+e)/(a+a*sin(f*x+e))**(1/2)/(c+d*sin(f*x+e))**(1/2),x)
```

output
```
Integral(1/(sqrt(a*(sin(e + f*x) + 1))*sqrt(c + d*sin(e + f*x))*sin(e + f*
x)), x)
```

3.38.7 Maxima [F]

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\,dx = \int \frac{1}{\sqrt{a\sin(fx+e)+a}\sqrt{d\sin(fx+e)+c}\sin(fx+e)}\,dx$$

input
```
integrate(1/sin(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x, al
gorithm="maxima")
```

output
```
integrate(1/(sqrt(a*sin(f*x + e) + a)*sqrt(d*sin(f*x + e) + c)*sin(f*x + e
)), x)
```

3.38.8 Giac [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\,dx = \text{Timed out}$$

input
```
integrate(1/sin(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x, al
gorithm="giac")
```

output
```
Timed out
```

3.38.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\,dx$$
$$=\int \frac{1}{\sin\left(e+f\,x\right)\,\sqrt{a+a\,\sin\left(e+f\,x\right)}\,\sqrt{c+d\,\sin\left(e+f\,x\right)}}\,dx$$

input
```
int(1/(sin(e + f*x)*(a + a*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))^(1/2))
,x)
```

output
```
int(1/(sin(e + f*x)*(a + a*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))^(1/2))
, x)
```

3.39 $\int \frac{\sin^2(e+fx)}{(a+b\sin(e+fx))^2(c+d\sin(e+fx))}\,dx$

3.39.1 Optimal result

Integrand size = 33, antiderivative size = 181

$$\int \frac{\sin^2(e+fx)}{(a+b\sin(e+fx))^2(c+d\sin(e+fx))}\,dx = -\frac{2a(a^2c-2b^2c+abd)\arctan\left(\frac{b+a\tan\left(\frac{1}{2}(e+fx)\right)}{\sqrt{a^2-b^2}}\right)}{(a^2-b^2)^{3/2}(bc-ad)^2f} + \frac{2c^2\arctan\left(\frac{d+c\tan\left(\frac{1}{2}(e+fx)\right)}{\sqrt{c^2-d^2}}\right)}{(bc-ad)^2\sqrt{c^2-d^2}f} + \frac{a^2\cos(e+fx)}{(a^2-b^2)(bc-ad)f(a+b\sin(e+fx))}$$

output
```
-2*a*(a^2*c+a*b*d-2*b^2*c)*arctan((b+a*tan(1/2*f*x+1/2*e))/(a^2-b^2)^(1/2)
)/(a^2-b^2)^(3/2)/(-a*d+b*c)^2/f+a^2*cos(f*x+e)/(a^2-b^2)/(-a*d+b*c)/f/(a+
b*sin(f*x+e))+2*c^2*arctan((d+c*tan(1/2*f*x+1/2*e))/(c^2-d^2)^(1/2))/(-a*d
+b*c)^2/f/(c^2-d^2)^(1/2)
```

3.39.2 Mathematica [A] (verified)

Time = 1.32 (sec) , antiderivative size = 178, normalized size of antiderivative = 0.98

$$\int \frac{\sin^2(e+fx)}{(a+b\sin(e+fx))^2(c+d\sin(e+fx))}\,dx = \frac{-\frac{2a(a^2c-2b^2c+abd)\arctan\left(\frac{b+a\tan\left(\frac{1}{2}(e+fx)\right)}{\sqrt{a^2-b^2}}\right)}{(a^2-b^2)^{3/2}(bc-ad)^2} + \frac{2c^2\arctan\left(\frac{d+c\tan\left(\frac{1}{2}(e+fx)\right)}{\sqrt{c^2-d^2}}\right)}{(bc-ad)^2\sqrt{c^2-d^2}} - \frac{a^2\cos(e+fx)}{(a-b)(a+b)(-bc+ad)(a+b\sin(e+fx))}}{f}$$

input
```
Integrate[Sin[e + f*x]^2/((a + b*Sin[e + f*x])^2*(c + d*Sin[e + f*x])),x]
```

output
```
((-2*a*(a^2*c - 2*b^2*c + a*b*d)*ArcTan[(b + a*Tan[(e + f*x)/2])/Sqrt[a^2
- b^2]])/((a^2 - b^2)^(3/2)*(b*c - a*d)^2) + (2*c^2*ArcTan[(d + c*Tan[(e +
 f*x)/2])/Sqrt[c^2 - d^2]])/((b*c - a*d)^2*Sqrt[c^2 - d^2]) - (a^2*Cos[e +
 f*x])/((a - b)*(a + b)*(-(b*c) + a*d)*(a + b*Sin[e + f*x])))/f
```

3.39.3 Rubi [A] (verified)

Time = 0.86 (sec) , antiderivative size = 225, normalized size of antiderivative = 1.24, number of steps used = 10, number of rules used = 9, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.273, Rules used = {3042, 3535, 25, 3042, 3480, 3042, 3139, 1083, 217}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sin^2(e+fx)}{(a+b\sin(e+fx))^2(c+d\sin(e+fx))}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\sin(e+fx)^2}{(a+b\sin(e+fx))^2(c+d\sin(e+fx))}dx$$

$$\Big\downarrow \text{3535}$$

$$\frac{a^2\cos(e+fx)}{f(a^2-b^2)(bc-ad)(a+b\sin(e+fx))}-\frac{\int -\frac{abc+(ca^2+bda-b^2c)\sin(e+fx)}{(a+b\sin(e+fx))(c+d\sin(e+fx))}dx}{(a^2-b^2)(bc-ad)}$$

$$\Big\downarrow \text{25}$$

$$\frac{\int \frac{abc+(ca^2+bda-b^2c)\sin(e+fx)}{(a+b\sin(e+fx))(c+d\sin(e+fx))}dx}{(a^2-b^2)(bc-ad)}+\frac{a^2\cos(e+fx)}{f(a^2-b^2)(bc-ad)(a+b\sin(e+fx))}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{abc+(ca^2+bda-b^2c)\sin(e+fx)}{(a+b\sin(e+fx))(c+d\sin(e+fx))}dx}{(a^2-b^2)(bc-ad)}+\frac{a^2\cos(e+fx)}{f(a^2-b^2)(bc-ad)(a+b\sin(e+fx))}$$

$$\Big\downarrow \text{3480}$$

$$\frac{\frac{c^2(a^2-b^2)\int \frac{1}{c+d\sin(e+fx)}dx}{bc-ad}-\frac{a(a^2c+abd-2b^2c)\int \frac{1}{a+b\sin(e+fx)}dx}{bc-ad}}{(a^2-b^2)(bc-ad)}+\frac{a^2\cos(e+fx)}{f(a^2-b^2)(bc-ad)(a+b\sin(e+fx))}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\frac{c^2(a^2-b^2)\int \frac{1}{c+d\sin(e+fx)}dx}{bc-ad}-\frac{a(a^2c+abd-2b^2c)\int \frac{1}{a+b\sin(e+fx)}dx}{bc-ad}}{(a^2-b^2)(bc-ad)}+\frac{a^2\cos(e+fx)}{f(a^2-b^2)(bc-ad)(a+b\sin(e+fx))}$$

$$\Big\downarrow \; 3139$$

$$\frac{\frac{2c^2(a^2-b^2)\int \frac{1}{c\tan^2\left(\frac{1}{2}(e+fx)\right)+2d\tan\left(\frac{1}{2}(e+fx)\right)+c}d\tan\left(\frac{1}{2}(e+fx)\right)}{f(bc-ad)}-\frac{2a(a^2c+abd-2b^2c)\int \frac{1}{a\tan^2\left(\frac{1}{2}(e+fx)\right)+2b\tan\left(\frac{1}{2}(e+fx)\right)+a}d\tan\left(\frac{1}{2}(e+fx)\right)}{f(bc-ad)}}{(a^2-b^2)(bc-ad)}$$
$$\frac{a^2\cos(e+fx)}{f(a^2-b^2)(bc-ad)(a+b\sin(e+fx))}$$

$$\Big\downarrow \; 1083$$

$$\frac{\frac{4a(a^2c+abd-2b^2c)\int \frac{1}{-\left(2b+2a\tan\left(\frac{1}{2}(e+fx)\right)\right)^2-4\left(a^2-b^2\right)}d\left(2b+2a\tan\left(\frac{1}{2}(e+fx)\right)\right)}{f(bc-ad)}-\frac{4c^2(a^2-b^2)\int \frac{1}{-\left(2d+2c\tan\left(\frac{1}{2}(e+fx)\right)\right)^2-4\left(c^2-d^2\right)}d(2d+2c}{f(bc-ad)}}{(a^2-b^2)(bc-ad)}$$
$$\frac{a^2\cos(e+fx)}{f(a^2-b^2)(bc-ad)(a+b\sin(e+fx))}$$

$$\Big\downarrow \; 217$$

$$\frac{\frac{2c^2(a^2-b^2)\arctan\left(\frac{2c\tan\left(\frac{1}{2}(e+fx)\right)+2d}{2\sqrt{c^2-d^2}}\right)}{f\sqrt{c^2-d^2}(bc-ad)}-\frac{2a(a^2c+abd-2b^2c)\arctan\left(\frac{2a\tan\left(\frac{1}{2}(e+fx)\right)+2b}{2\sqrt{a^2-b^2}}\right)}{f\sqrt{a^2-b^2}(bc-ad)}}{(a^2-b^2)(bc-ad)}+$$
$$\frac{a^2\cos(e+fx)}{f(a^2-b^2)(bc-ad)(a+b\sin(e+fx))}$$

input
```
Int[Sin[e + f*x]^2/((a + b*Sin[e + f*x])^2*(c + d*Sin[e + f*x])),x]
```

output
```
((-2*a*(a^2*c - 2*b^2*c + a*b*d)*ArcTan[(2*b + 2*a*Tan[(e + f*x)/2])/(2*Sq
rt[a^2 - b^2])])/(Sqrt[a^2 - b^2]*(b*c - a*d)*f) + (2*(a^2 - b^2)*c^2*ArcT
an[(2*d + 2*c*Tan[(e + f*x)/2])/(2*Sqrt[c^2 - d^2])])/((b*c - a*d)*Sqrt[c^
2 - d^2]*f))/((a^2 - b^2)*(b*c - a*d)) + (a^2*Cos[e + f*x])/((a^2 - b^2)*(
b*c - a*d)*f*(a + b*Sin[e + f*x]))
```

3.39. $\int \frac{\sin^2(e+fx)}{(a+b\sin(e+fx))^2(c+d\sin(e+fx))}\,dx$

3.39.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 217
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3139
```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Simp[2*(e/d)   Subst[Int[1/(a + 2*b*e*x + a
*e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[a^2 - b^2, 0]
```

rule 3480
```
Int[((A_.) + (B_.)*sin[(e_.) + (f_.)*(x_)])/(((a_.) + (b_.)*sin[(e_.) + (f_
.)*(x_)])*((c_.) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[(A*b
- a*B)/(b*c - a*d)   Int[1/(a + b*Sin[e + f*x]), x], x] + Simp[(B*c - A*d)/
(b*c - a*d)   Int[1/(c + d*Sin[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f
, A, B}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]
```

rule 3535
```
Int[((a_.) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_)*((c_.) + (d_.)*sin[(e_.) +
 (f_.)*(x_)])^(n_)*((A_.) + (C_.)*sin[(e_.) + (f_.)*(x_)]^2), x_Symbol] :>
Simp[(-(A*b^2 + a^2*C))*Cos[e + f*x]*(a + b*Sin[e + f*x])^(m + 1)*((c + d*S
in[e + f*x])^(n + 1)/(f*(m + 1)*(b*c - a*d)*(a^2 - b^2))), x] + Simp[1/((m
+ 1)*(b*c - a*d)*(a^2 - b^2))   Int[(a + b*Sin[e + f*x])^(m + 1)*(c + d*Sin
[e + f*x])^n*Simp[a*(m + 1)*(b*c - a*d)*(A + C) + d*(A*b^2 + a^2*C)*(m + n
+ 2) - (c*(A*b^2 + a^2*C) + b*(m + 1)*(b*c - a*d)*(A + C))*Sin[e + f*x] - d
*(A*b^2 + a^2*C)*(m + n + 3)*Sin[e + f*x]^2, x], x], x] /; FreeQ[{a, b, c,
d, e, f, A, C, n}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2
- d^2, 0] && LtQ[m, -1] && ((EqQ[a, 0] && IntegerQ[m] &&  !IntegerQ[n]) ||
 !(IntegerQ[2*n] && LtQ[n, -1] && ((IntegerQ[n] &&  !IntegerQ[m]) || EqQ[a,
 0])))
```

3.39.4 Maple [A] (verified)

Time = 2.03 (sec) , antiderivative size = 244, normalized size of antiderivative = 1.35

method	result
derivativedivides	$\frac{-\frac{2a\left(\frac{\frac{b(ad-bc)\tan\left(\frac{fx}{2}+\frac{e}{2}\right)}{a^2-b^2}+\frac{a(ad-bc)}{a^2-b^2}}{\left(\tan^2\left(\frac{fx}{2}+\frac{e}{2}\right)\right)a+2b\tan\left(\frac{fx}{2}+\frac{e}{2}\right)+a}+\frac{\left(a^2c+abd-2b^2c\right)\arctan\left(\frac{2a\tan\left(\frac{fx}{2}+\frac{e}{2}\right)+2b}{2\sqrt{a^2-b^2}}\right)}{\left(a^2-b^2\right)^{\frac{3}{2}}}\right)}{a^2d^2-2abcd+b^2c^2}+\frac{8c^2\arctan\left(\frac{2c\tan\left(\frac{fx}{2}+\frac{e}{2}\right)+2d}{2\sqrt{c^2-d^2}}\right)}{\left(4a^2d^2-8abcd+4b^2c^2\right)\sqrt{c^2-d^2}}}{f}$
default	$\frac{-\frac{2a\left(\frac{\frac{b(ad-bc)\tan\left(\frac{fx}{2}+\frac{e}{2}\right)}{a^2-b^2}+\frac{a(ad-bc)}{a^2-b^2}}{\left(\tan^2\left(\frac{fx}{2}+\frac{e}{2}\right)\right)a+2b\tan\left(\frac{fx}{2}+\frac{e}{2}\right)+a}+\frac{\left(a^2c+abd-2b^2c\right)\arctan\left(\frac{2a\tan\left(\frac{fx}{2}+\frac{e}{2}\right)+2b}{2\sqrt{a^2-b^2}}\right)}{\left(a^2-b^2\right)^{\frac{3}{2}}}\right)}{a^2d^2-2abcd+b^2c^2}+\frac{8c^2\arctan\left(\frac{2c\tan\left(\frac{fx}{2}+\frac{e}{2}\right)+2d}{2\sqrt{c^2-d^2}}\right)}{\left(4a^2d^2-8abcd+4b^2c^2\right)\sqrt{c^2-d^2}}}{f}$
risch	$\frac{2ia^2\left(ib+a\,e^{i(fx+e)}\right)}{b(a^2-b^2)(ad-bc)f\left(-ie^{2i(fx+e)}b+2a\,e^{i(fx+e)}+ib\right)}+\frac{c^2\ln\left(e^{i(fx+e)}+\frac{ic\sqrt{-c^2+d^2}+c^2-d^2}{\sqrt{-c^2+d^2}\,d}\right)}{\sqrt{-c^2+d^2}\,(ad-bc)^2f}-\frac{c^2\ln\left(e^{i(fx+e)}+\frac{ic\sqrt{-\cdots}}{\cdots}\right)}{\sqrt{-c^2+d^2}\,(a\cdots}$

input
```
int(sin(f*x+e)^2/(a+b*sin(f*x+e))^2/(c+d*sin(f*x+e)),x,method=_RETURNVERBO
SE)
```

output
```
1/f*(-2*a/(a^2*d^2-2*a*b*c*d+b^2*c^2)*((b*(a*d-b*c)/(a^2-b^2)*tan(1/2*f*x+
1/2*e)+a*(a*d-b*c)/(a^2-b^2))/(tan(1/2*f*x+1/2*e)^2*a+2*b*tan(1/2*f*x+1/2*
e)+a)+(a^2*c+a*b*d-2*b^2*c)/(a^2-b^2)^(3/2)*arctan(1/2*(2*a*tan(1/2*f*x+1/
2*e)+2*b)/(a^2-b^2)^(1/2)))+8*c^2/(4*a^2*d^2-8*a*b*c*d+4*b^2*c^2)/(c^2-d^2
)^(1/2)*arctan(1/2*(2*c*tan(1/2*f*x+1/2*e)+2*d)/(c^2-d^2)^(1/2)))
```

3.39.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 625 vs. $2(171) = 342$.

Time = 65.07 (sec) , antiderivative size = 2837, normalized size of antiderivative = 15.67

$$\int \frac{\sin^2(e+fx)}{(a+b\sin(e+fx))^2(c+d\sin(e+fx))}\,dx = \text{Too large to display}$$

input
```
integrate(sin(f*x+e)^2/(a+b*sin(f*x+e))^2/(c+d*sin(f*x+e)),x, algorithm="f
ricas")
```

output
```
[1/2*((a^3*b*c^2*d - a^3*b*d^3 + (a^4 - 2*a^2*b^2)*c^3 - (a^4 - 2*a^2*b^2)
*c*d^2 + (a^2*b^2*c^2*d - a^2*b^2*d^3 + (a^3*b - 2*a*b^3)*c^3 - (a^3*b - 2
*a*b^3)*c*d^2)*sin(f*x + e))*sqrt(-a^2 + b^2)*log(((2*a^2 - b^2)*cos(f*x +
 e)^2 - 2*a*b*sin(f*x + e) - a^2 - b^2 + 2*(a*cos(f*x + e)*sin(f*x + e) +
b*cos(f*x + e))*sqrt(-a^2 + b^2))/(b^2*cos(f*x + e)^2 - 2*a*b*sin(f*x + e)
 - a^2 - b^2)) - ((a^4*b - 2*a^2*b^3 + b^5)*c^2*sin(f*x + e) + (a^5 - 2*a^
3*b^2 + a*b^4)*c^2)*sqrt(-c^2 + d^2)*log(((2*c^2 - d^2)*cos(f*x + e)^2 - 2
*c*d*sin(f*x + e) - c^2 - d^2 + 2*(c*cos(f*x + e)*sin(f*x + e) + d*cos(f*x
 + e))*sqrt(-c^2 + d^2))/(d^2*cos(f*x + e)^2 - 2*c*d*sin(f*x + e) - c^2 -
d^2)) + 2*((a^4*b - a^2*b^3)*c^3 - (a^5 - a^3*b^2)*c^2*d - (a^4*b - a^2*b^
3)*c*d^2 + (a^5 - a^3*b^2)*d^3)*cos(f*x + e))/(((a^4*b^3 - 2*a^2*b^5 + b^7
)*c^4 - 2*(a^5*b^2 - 2*a^3*b^4 + a*b^6)*c^3*d + (a^6*b - 3*a^4*b^3 + 3*a^2
*b^5 - b^7)*c^2*d^2 + 2*(a^5*b^2 - 2*a^3*b^4 + a*b^6)*c*d^3 - (a^6*b - 2*a
^4*b^3 + a^2*b^5)*d^4)*f*sin(f*x + e) + ((a^5*b^2 - 2*a^3*b^4 + a*b^6)*c^4
 - 2*(a^6*b - 2*a^4*b^3 + a^2*b^5)*c^3*d + (a^7 - 3*a^5*b^2 + 3*a^3*b^4 -
a*b^6)*c^2*d^2 + 2*(a^6*b - 2*a^4*b^3 + a^2*b^5)*c*d^3 - (a^7 - 2*a^5*b^2
+ a^3*b^4)*d^4)*f), -1/2*(2*((a^4*b - 2*a^2*b^3 + b^5)*c^2*sin(f*x + e) +
(a^5 - 2*a^3*b^2 + a*b^4)*c^2)*sqrt(c^2 - d^2)*arctan(-(c*sin(f*x + e) + d
)/(sqrt(c^2 - d^2)*cos(f*x + e))) - (a^3*b*c^2*d - a^3*b*d^3 + (a^4 - 2*a^
2*b^2)*c^3 - (a^4 - 2*a^2*b^2)*c*d^2 + (a^2*b^2*c^2*d - a^2*b^2*d^3 + (...
```

3.39.6 Sympy [F(-1)]

Timed out.

$$\int \frac{\sin^2(e+fx)}{(a+b\sin(e+fx))^2(c+d\sin(e+fx))}\,dx = \text{Timed out}$$

input
```
integrate(sin(f*x+e)**2/(a+b*sin(f*x+e))**2/(c+d*sin(f*x+e)),x)
```

output
```
Timed out
```

3.39.7 Maxima [F(-2)]

Exception generated.

$$\int \frac{\sin^2(e+fx)}{(a+b\sin(e+fx))^2(c+d\sin(e+fx))}\,dx = \text{Exception raised: ValueError}$$

input
```
integrate(sin(f*x+e)^2/(a+b*sin(f*x+e))^2/(c+d*sin(f*x+e)),x, algorithm="m
axima")
```

output
```
Exception raised: ValueError >> Computation failed since Maxima requested
additional constraints; using the 'assume' command before evaluation *may*
 help (example of legal syntax is 'assume(4*d^2-4*c^2>0)', see `assume?` f
or more de
```

3.39.8 Giac [A] (verification not implemented)

Time = 0.44 (sec) , antiderivative size = 292, normalized size of antiderivative = 1.61

$$\int \frac{\sin^2(e+fx)}{(a+b\sin(e+fx))^2(c+d\sin(e+fx))}\,dx = \frac{2\left(\frac{\left(\pi\left\lfloor\frac{fx+e}{2\pi}+\frac{1}{2}\right\rfloor\operatorname{sgn}(c)+\arctan\left(\frac{c\tan\left(\frac{1}{2}fx+\frac{1}{2}e\right)+d}{\sqrt{c^2-d^2}}\right)\right)c^2}{(b^2c^2-2abcd+a^2d^2)\sqrt{c^2-d^2}} - \frac{(a^3c-2ab^2c+a^2bd)\left(\pi\left\lfloor\frac{fx+e}{2\pi}+\frac{1}{2}\right\rfloor\operatorname{sgn}(a)+\arctan\left(\frac{a\tan\left(\frac{1}{2}fx+\frac{1}{2}e\right)+b}{\sqrt{a^2-b^2}}\right)\right)}{(a^2b^2c^2-b^4c^2-2a^3bcd+2ab^3cd+a^4d^2-a^2b^2d^2)\sqrt{a^2-b^2}} + \right.}{f}$$

input
```
integrate(sin(f*x+e)^2/(a+b*sin(f*x+e))^2/(c+d*sin(f*x+e)),x, algorithm="g
iac")
```

output
```
2*((pi*floor(1/2*(f*x + e)/pi + 1/2)*sgn(c) + arctan((c*tan(1/2*f*x + 1/2*
e) + d)/sqrt(c^2 - d^2)))*c^2/((b^2*c^2 - 2*a*b*c*d + a^2*d^2)*sqrt(c^2 -
d^2)) - (a^3*c - 2*a*b^2*c + a^2*b*d)*(pi*floor(1/2*(f*x + e)/pi + 1/2)*sg
n(a) + arctan((a*tan(1/2*f*x + 1/2*e) + b)/sqrt(a^2 - b^2)))/((a^2*b^2*c^2
 - b^4*c^2 - 2*a^3*b*c*d + 2*a*b^3*c*d + a^4*d^2 - a^2*b^2*d^2)*sqrt(a^2 -
 b^2)) + (a*b*tan(1/2*f*x + 1/2*e) + a^2)/((a^2*b*c - b^3*c - a^3*d + a*b^
2*d)*(a*tan(1/2*f*x + 1/2*e)^2 + 2*b*tan(1/2*f*x + 1/2*e) + a))/f
```

3.39.9 Mupad [B] (verification not implemented)

Time = 27.41 (sec) , antiderivative size = 23933, normalized size of antiderivative = 132.23

$$\int \frac{\sin^2(e+fx)}{(a+b\sin(e+fx))^2(c+d\sin(e+fx))}\,dx = \text{Too large to display}$$

input
```
int(sin(e + f*x)^2/((a + b*sin(e + f*x))^2*(c + d*sin(e + f*x))),x)
```

output
```
- ((2*a^2)/((a^2 - b^2)*(a*d - b*c)) + (2*a*b*tan(e/2 + (f*x)/2))/((a^2 -
b^2)*(a*d - b*c)))/(f*(a + 2*b*tan(e/2 + (f*x)/2) + a*tan(e/2 + (f*x)/2)^2
)) - (c^2*atan(((c^2*(d^2 - c^2)^(1/2)*((32*(2*a^4*b^4*c^6 - a^2*b^6*c^6 -
 a^6*b^2*c^6 + a^8*c^4*d^2 - 3*a^3*b^5*c^5*d + 2*a^5*b^3*c^5*d + 2*a^7*b*c
^3*d^3 + 8*a^4*b^4*c^4*d^2 - 5*a^5*b^3*c^3*d^3 + a^6*b^2*c^2*d^4 - 6*a^6*b
^2*c^4*d^2))/(a^7*d^3 - b^7*c^3 + 2*a^2*b^5*c^3 - a^4*b^3*c^3 + a^3*b^4*d^
3 - 2*a^5*b^2*d^3 - 3*a^2*b^5*c*d^2 - 6*a^3*b^4*c^2*d + 6*a^4*b^3*c*d^2 +
3*a^5*b^2*c^2*d + 3*a*b^6*c^2*d - 3*a^6*b*c*d^2) + (32*tan(e/2 + (f*x)/2)*
(2*a^7*b*c^6 - 2*a*b^7*c^6 - 2*a^8*c^5*d + 9*a^3*b^5*c^6 - 8*a^5*b^3*c^6 +
 2*a^8*c^3*d^3 + 2*a^2*b^6*c^5*d - 13*a^4*b^4*c^5*d + 2*a^6*b^2*c*d^5 + 10
*a^6*b^2*c^5*d + 4*a^7*b*c^2*d^4 - 4*a^7*b*c^4*d^2 - 8*a^3*b^5*c^4*d^2 + 1
6*a^4*b^4*c^3*d^3 - 10*a^5*b^3*c^2*d^4 + 13*a^5*b^3*c^4*d^2 - 13*a^6*b^2*c
^3*d^3))/(a^7*d^3 - b^7*c^3 + 2*a^2*b^5*c^3 - a^4*b^3*c^3 + a^3*b^4*d^3 -
2*a^5*b^2*d^3 - 3*a^2*b^5*c*d^2 - 6*a^3*b^4*c^2*d + 6*a^4*b^3*c*d^2 + 3*a^
5*b^2*c^2*d + 3*a*b^6*c^2*d - 3*a^6*b*c*d^2) + (c^2*(d^2 - c^2)^(1/2)*((32
*(a*b^9*c^7 - a^3*b^7*c^7 + a^10*c^2*d^5 - 3*a^2*b^8*c^6*d + 2*a^4*b^6*c^6
*d + a^6*b^4*c^6*d - a^7*b^3*c*d^6 - 4*a^9*b*c^3*d^4 + a^3*b^7*c^5*d^2 + 6
*a^4*b^6*c^4*d^3 - 9*a^5*b^5*c^3*d^4 + 3*a^5*b^5*c^5*d^2 + 5*a^6*b^4*c^2*d
^5 - 12*a^6*b^4*c^4*d^3 + 13*a^7*b^3*c^3*d^4 - 4*a^7*b^3*c^5*d^2 - 6*a^8*b
^2*c^2*d^5 + 6*a^8*b^2*c^4*d^3 + a^9*b*c*d^6))/(a^7*d^3 - b^7*c^3 + 2*a...
```

3.40 $\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{a+b\sin(e+fx)}\,dx$

3.40.1 Optimal result

Integrand size = 33, antiderivative size = 154

$$\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{a+b\sin(e+fx)}\,dx = \frac{2c\,\text{EllipticPi}\left(2,\frac{1}{2}\left(e-\frac{\pi}{2}+fx\right),\frac{2d}{c+d}\right)\sqrt{\frac{c+d\sin(e+fx)}{c+d}}}{af\sqrt{c+d\sin(e+fx)}} - \frac{2(bc-ad)\,\text{EllipticPi}\left(\frac{2b}{a+b},\frac{1}{2}\left(e-\frac{\pi}{2}+fx\right),\frac{2d}{c+d}\right)\sqrt{\frac{c+d\sin(e+fx)}{c+d}}}{a(a+b)f\sqrt{c+d\sin(e+fx)}}$$

output
```
-2*c*(sin(1/2*e+1/4*Pi+1/2*f*x)^2)^(1/2)/sin(1/2*e+1/4*Pi+1/2*f*x)*Ellipti
cPi(cos(1/2*e+1/4*Pi+1/2*f*x),2,2^(1/2)*(d/(c+d))^(1/2))*((c+d*sin(f*x+e))
/(c+d))^(1/2)/a/f/(c+d*sin(f*x+e))^(1/2)+2*(-a*d+b*c)*(sin(1/2*e+1/4*Pi+1/
2*f*x)^2)^(1/2)/sin(1/2*e+1/4*Pi+1/2*f*x)*EllipticPi(cos(1/2*e+1/4*Pi+1/2*
f*x),2*b/(a+b),2^(1/2)*(d/(c+d))^(1/2))*((c+d*sin(f*x+e))/(c+d))^(1/2)/a/(
a+b)/f/(c+d*sin(f*x+e))^(1/2)
```

3.40.2 Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 30.02 (sec) , antiderivative size = 179, normalized size of antiderivative = 1.16

$$\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{a+b\sin(e+fx)}\,dx = \frac{2i\left(\text{EllipticPi}\left(\frac{c+d}{c}, i\text{arcsinh}\left(\sqrt{-\frac{1}{c+d}}\sqrt{c+d\sin(e+fx)}\right), \frac{c+d}{c-d}\right) - \text{EllipticPi}\left(\frac{b(c+d)}{bc-ad}, i\text{arcsinh}\left(\sqrt{-\frac{1}{c+d}}\sqrt{\right.\right.\right.}{a\sqrt{-\frac{1}{c+d}}f}$$

input
```
Integrate[(Csc[e + f*x]*Sqrt[c + d*Sin[e + f*x]])/(a + b*Sin[e + f*x]),x]
```

output
```
((2*I)*(EllipticPi[(c + d)/c, I*ArcSinh[Sqrt[-(c + d)^(-1)]*Sqrt[c + d*Sin
[e + f*x]]], (c + d)/(c - d)] - EllipticPi[(b*(c + d))/(b*c - a*d), I*ArcS
inh[Sqrt[-(c + d)^(-1)]*Sqrt[c + d*Sin[e + f*x]]], (c + d)/(c - d)])*Sec[e
 + f*x]*Sqrt[-((d*(-1 + Sin[e + f*x]))/(c + d))]*Sqrt[(d*(1 + Sin[e + f*x]
))/(-c + d)])/(a*Sqrt[-(c + d)^(-1)]*f)
```

3.40.3 Rubi [A] (verified)

Time = 0.83 (sec) , antiderivative size = 154, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 6, $\frac{\text{number of rules}}{\text{integrand size}} = 0.182$, Rules used = {3042, 3414, 3042, 3286, 3042, 3284}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{a+b\sin(e+fx)}\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\sqrt{c+d\sin(e+fx)}}{\sin(e+fx)(a+b\sin(e+fx))}dx$$

$$\Big\downarrow \text{3414}$$

$$\frac{c\int \frac{\csc(e+fx)}{\sqrt{c+d\sin(e+fx)}}dx}{a} - \frac{(bc-ad)\int \frac{1}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}dx}{a}$$

$$\Big\downarrow \text{3042}$$

$$\frac{c\int \frac{1}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}}dx}{a}-\frac{(bc-ad)\int \frac{1}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}dx}{a}$$

$$\Big\downarrow \text{3286}$$

$$\frac{c\sqrt{\frac{c+d\sin(e+fx)}{c+d}}\int \frac{\csc(e+fx)}{\sqrt{\frac{c}{c+d}+\frac{d\sin(e+fx)}{c+d}}}dx}{a\sqrt{c+d\sin(e+fx)}}-\frac{(bc-ad)\sqrt{\frac{c+d\sin(e+fx)}{c+d}}\int \frac{1}{(a+b\sin(e+fx))\sqrt{\frac{c}{c+d}+\frac{d\sin(e+fx)}{c+d}}}dx}{a\sqrt{c+d\sin(e+fx)}}$$

$$\Big\downarrow \text{3042}$$

$$\frac{c\sqrt{\frac{c+d\sin(e+fx)}{c+d}}\int \frac{1}{\sin(e+fx)\sqrt{\frac{c}{c+d}+\frac{d\sin(e+fx)}{c+d}}}dx}{a\sqrt{c+d\sin(e+fx)}}-\frac{(bc-ad)\sqrt{\frac{c+d\sin(e+fx)}{c+d}}\int \frac{1}{(a+b\sin(e+fx))\sqrt{\frac{c}{c+d}+\frac{d\sin(e+fx)}{c+d}}}dx}{a\sqrt{c+d\sin(e+fx)}}$$

$$\Big\downarrow \text{3284}$$

$$\frac{2c\sqrt{\frac{c+d\sin(e+fx)}{c+d}}\text{EllipticPi}\left(2,\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right),\frac{2d}{c+d}\right)}{af\sqrt{c+d\sin(e+fx)}}-\frac{2(bc-ad)\sqrt{\frac{c+d\sin(e+fx)}{c+d}}\text{EllipticPi}\left(\frac{2b}{a+b},\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right),\frac{2d}{c+d}\right)}{af(a+b)\sqrt{c+d\sin(e+fx)}}$$

input
```
Int[(Csc[e + f*x]*Sqrt[c + d*Sin[e + f*x]])/(a + b*Sin[e + f*x]),x]
```

output
```
(2*c*EllipticPi[2, (e - Pi/2 + f*x)/2, (2*d)/(c + d)]*Sqrt[(c + d*Sin[e +
f*x])/(c + d)])/(a*f*Sqrt[c + d*Sin[e + f*x]]) - (2*(b*c - a*d)*EllipticPi
[(2*b)/(a + b), (e - Pi/2 + f*x)/2, (2*d)/(c + d)]*Sqrt[(c + d*Sin[e + f*x
])/(c + d)])/(a*(a + b)*f*Sqrt[c + d*Sin[e + f*x]])
```

3.40. $\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{a+b\sin(e+fx)}\,dx$

3.40.3.1 Defintions of rubi rules used

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3284
```
Int[1/(((a_.) + (b_.)*sin[(e_.) + (f_.)*(x_)])*Sqrt[(c_.) + (d_.)*sin[(e_.)
 + (f_.)*(x_)]]), x_Symbol] :> Simp[(2/(f*(a + b)*Sqrt[c + d]))*EllipticPi[
2*(b/(a + b)), (1/2)*(e - Pi/2 + f*x), 2*(d/(c + d))], x] /; FreeQ[{a, b, c
, d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2,
0] && GtQ[c + d, 0]
```

rule 3286
```
Int[1/(((a_.) + (b_.)*sin[(e_.) + (f_.)*(x_)])*Sqrt[(c_.) + (d_.)*sin[(e_.)
 + (f_.)*(x_)]]), x_Symbol] :> Simp[Sqrt[(c + d*Sin[e + f*x])/(c + d)]/Sqrt
[c + d*Sin[e + f*x]]   Int[1/((a + b*Sin[e + f*x])*Sqrt[c/(c + d) + (d/(c +
 d))*Sin[e + f*x]]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*
d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0] &&  !GtQ[c + d, 0]
```

rule 3414
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(sin[(e_.) + (f_.)*(x_)]*((c
_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[a/c   Int[1/(Sin[e
+ f*x]*Sqrt[a + b*Sin[e + f*x]]), x], x] + Simp[(b*c - a*d)/c   Int[1/(Sqrt
[a + b*Sin[e + f*x]]*(c + d*Sin[e + f*x])), x], x] /; FreeQ[{a, b, c, d, e,
 f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0]
```

3.40.4 Maple [A] (verified)

Time = 1.29 (sec) , antiderivative size = 190, normalized size of antiderivative = 1.23

method	result
default	$-\frac{2\left(\Pi\left(\sqrt{\frac{c+d\sin(fx+e)}{c-d}},\frac{c-d}{c},\sqrt{\frac{c-d}{c+d}}\right)-\Pi\left(\sqrt{\frac{c+d\sin(fx+e)}{c-d}},-\frac{(c-d)b}{ad-bc},\sqrt{\frac{c-d}{c+d}}\right)\right)\sqrt{-\frac{d(1+\sin(fx+e))}{c-d}}\,\sqrt{-\frac{(\sin(fx+e)-1)d}{c+d}}\,\sqrt{\frac{c+d\sin(fx+e)}{c-d}}}{a\cos(fx+e)\sqrt{c+d\sin(fx+e)}\,f}$

input
```
int((c+d*sin(f*x+e))^(1/2)/sin(f*x+e)/(a+b*sin(f*x+e)),x,method=_RETURNVER
BOSE)
```

output
```
-2*(EllipticPi(((c+d*sin(f*x+e))/(c-d))^(1/2),(c-d)/c,((c-d)/(c+d))^(1/2))
-EllipticPi(((c+d*sin(f*x+e))/(c-d))^(1/2),-(c-d)*b/(a*d-b*c),((c-d)/(c+d)
)^(1/2)))*(-d*(1+sin(f*x+e))/(c-d))^(1/2)*(-(sin(f*x+e)-1)*d/(c+d))^(1/2)*
((c+d*sin(f*x+e))/(c-d))^(1/2)*(c-d)/a/cos(f*x+e)/(c+d*sin(f*x+e))^(1/2)/f
```

3.40.5 Fricas [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{a+b\sin(e+fx)}\,dx = \text{Timed out}$$

input
```
integrate((c+d*sin(f*x+e))^(1/2)/sin(f*x+e)/(a+b*sin(f*x+e)),x, algorithm=
"fricas")
```

output
```
Timed out
```

3.40.6 Sympy [F]

$$\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{a+b\sin(e+fx)}\,dx = \int \frac{\sqrt{c+d\sin(e+fx)}}{(a+b\sin(e+fx))\sin(e+fx)}\,dx$$

input
```
integrate((c+d*sin(f*x+e))**(1/2)/sin(f*x+e)/(a+b*sin(f*x+e)),x)
```

output
```
Integral(sqrt(c + d*sin(e + f*x))/((a + b*sin(e + f*x))*sin(e + f*x)), x)
```

3.40.7 Maxima [F]

$$\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{a+b\sin(e+fx)}\,dx = \int \frac{\sqrt{d\sin(fx+e)+c}}{(b\sin(fx+e)+a)\sin(fx+e)}\,dx$$

input
```
integrate((c+d*sin(f*x+e))^(1/2)/sin(f*x+e)/(a+b*sin(f*x+e)),x, algorithm=
"maxima")
```

output
```
integrate(sqrt(d*sin(f*x + e) + c)/((b*sin(f*x + e) + a)*sin(f*x + e)), x)
```

3.40.8 Giac [F]

$$\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{a+b\sin(e+fx)}\,dx = \int \frac{\sqrt{d\sin(fx+e)+c}}{(b\sin(fx+e)+a)\sin(fx+e)}\,dx$$

input
```
integrate((c+d*sin(f*x+e))^(1/2)/sin(f*x+e)/(a+b*sin(f*x+e)),x, algorithm=
"giac")
```

output
```
integrate(sqrt(d*sin(f*x + e) + c)/((b*sin(f*x + e) + a)*sin(f*x + e)), x)
```

3.40.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{a+b\sin(e+fx)}\,dx = \int \frac{\sqrt{c+d\,\sin(e+f\,x)}}{\sin(e+f\,x)\,\left(a+b\,\sin(e+f\,x)\right)}\,dx$$

input
```
int((c + d*sin(e + f*x))^(1/2)/(sin(e + f*x)*(a + b*sin(e + f*x))),x)
```

output
```
int((c + d*sin(e + f*x))^(1/2)/(sin(e + f*x)*(a + b*sin(e + f*x))), x)
```

3.41 $\int \frac{\csc(e+fx)}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}\,dx$

3.41.1 Optimal result

Integrand size = 33, antiderivative size = 146

$$\int \frac{\csc(e+fx)}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}\,dx = \frac{2\,\text{EllipticPi}\left(2,\frac{1}{2}\left(e-\frac{\pi}{2}+fx\right),\frac{2d}{c+d}\right)\sqrt{\frac{c+d\sin(e+fx)}{c+d}}}{af\sqrt{c+d\sin(e+fx)}} - \frac{2b\,\text{EllipticPi}\left(\frac{2b}{a+b},\frac{1}{2}\left(e-\frac{\pi}{2}+fx\right),\frac{2d}{c+d}\right)\sqrt{\frac{c+d\sin(e+fx)}{c+d}}}{a(a+b)f\sqrt{c+d\sin(e+fx)}}$$

output
```
-2*(sin(1/2*e+1/4*Pi+1/2*f*x)^2)^(1/2)/sin(1/2*e+1/4*Pi+1/2*f*x)*EllipticP
i(cos(1/2*e+1/4*Pi+1/2*f*x),2,2^(1/2)*(d/(c+d))^(1/2))*((c+d*sin(f*x+e))/(
c+d))^(1/2)/a/f/(c+d*sin(f*x+e))^(1/2)+2*b*(sin(1/2*e+1/4*Pi+1/2*f*x)^2)^(
1/2)/sin(1/2*e+1/4*Pi+1/2*f*x)*EllipticPi(cos(1/2*e+1/4*Pi+1/2*f*x),2*b/(a
+b),2^(1/2)*(d/(c+d))^(1/2))*((c+d*sin(f*x+e))/(c+d))^(1/2)/a/(a+b)/f/(c+d
*sin(f*x+e))^(1/2)
```

3.41.2 Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 29.59 (sec) , antiderivative size = 203, normalized size of antiderivative = 1.39

$$\int \frac{\csc(e+fx)}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}\,dx = -\frac{2i\left((-bc+ad)\,\text{EllipticPi}\left(\frac{c+d}{c},i\text{arcsinh}\left(\sqrt{-\frac{1}{c+d}}\sqrt{c+d\sin(e+fx)}\right),\frac{c+d}{c-d}\right)+bc\,\text{EllipticPi}\left(\frac{b(c+d)}{bc-ad},i\text{arc}\right.\right.}{ac\sqrt{-\frac{1}{c+d}}(bc-a}$$

input
```
Integrate[Csc[e + f*x]/((a + b*Sin[e + f*x])*Sqrt[c + d*Sin[e + f*x]]),x]
```

output
```
((-2*I)*((-(b*c) + a*d)*EllipticPi[(c + d)/c, I*ArcSinh[Sqrt[-(c + d)^(-1)
]*Sqrt[c + d*Sin[e + f*x]]], (c + d)/(c - d)] + b*c*EllipticPi[(b*(c + d))
/(b*c - a*d), I*ArcSinh[Sqrt[-(c + d)^(-1)]*Sqrt[c + d*Sin[e + f*x]]], (c
+ d)/(c - d)])*Sec[e + f*x]*Sqrt[-((d*(-1 + Sin[e + f*x]))/(c + d))]*Sqrt[
-((d*(1 + Sin[e + f*x]))/(c - d))])/(a*c*Sqrt[-(c + d)^(-1)]*(b*c - a*d)*f
)
```

3.41.3 Rubi [A] (verified)

Time = 0.85 (sec) , antiderivative size = 146, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 6, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.182, Rules used = {3042, 3420, 3042, 3286, 3042, 3284}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(e+fx)}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{1}{\sin(e+fx)(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}dx$$

$$\Big\downarrow \text{3420}$$

$$\frac{\int \frac{\csc(e+fx)}{\sqrt{c+d\sin(e+fx)}}dx}{a}-\frac{b\int \frac{1}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}dx}{a}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{1}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}}dx}{a}-\frac{b\int \frac{1}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}dx}{a}$$

$$\Big\downarrow \text{3286}$$

$$\frac{\sqrt{\frac{c+d\sin(e+fx)}{c+d}}\int \frac{\csc(e+fx)}{\sqrt{\frac{c}{c+d}+\frac{d\sin(e+fx)}{c+d}}}dx}{a\sqrt{c+d\sin(e+fx)}}-\frac{b\sqrt{\frac{c+d\sin(e+fx)}{c+d}}\int \frac{1}{(a+b\sin(e+fx))\sqrt{\frac{c}{c+d}+\frac{d\sin(e+fx)}{c+d}}}dx}{a\sqrt{c+d\sin(e+fx)}}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\sqrt{\frac{c+d\sin(e+fx)}{c+d}}\int \frac{1}{\sin(e+fx)\sqrt{\frac{c}{c+d}+\frac{d\sin(e+fx)}{c+d}}}dx}{a\sqrt{c+d\sin(e+fx)}}-\frac{b\sqrt{\frac{c+d\sin(e+fx)}{c+d}}\int \frac{1}{(a+b\sin(e+fx))\sqrt{\frac{c}{c+d}+\frac{d\sin(e+fx)}{c+d}}}dx}{a\sqrt{c+d\sin(e+fx)}}$$

$$\Big\downarrow \text{3284}$$

$$\frac{2\sqrt{\frac{c+d\sin(e+fx)}{c+d}}\text{EllipticPi}\left(2,\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right),\frac{2d}{c+d}\right)}{af\sqrt{c+d\sin(e+fx)}}-\frac{2b\sqrt{\frac{c+d\sin(e+fx)}{c+d}}\text{EllipticPi}\left(\frac{2b}{a+b},\frac{1}{2}\left(e+fx-\frac{\pi}{2}\right),\frac{2d}{c+d}\right)}{af(a+b)\sqrt{c+d\sin(e+fx)}}$$

input
```
Int[Csc[e + f*x]/((a + b*Sin[e + f*x])*Sqrt[c + d*Sin[e + f*x]]),x]
```

output
```
(2*EllipticPi[2, (e - Pi/2 + f*x)/2, (2*d)/(c + d)]*Sqrt[(c + d*Sin[e + f*
x])/(c + d)])/(a*f*Sqrt[c + d*Sin[e + f*x]]) - (2*b*EllipticPi[(2*b)/(a +
b), (e - Pi/2 + f*x)/2, (2*d)/(c + d)]*Sqrt[(c + d*Sin[e + f*x])/(c + d)])
/(a*(a + b)*f*Sqrt[c + d*Sin[e + f*x]])
```

3.41. $\int \frac{\csc(e+fx)}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}\,dx$

3.41.3.1 Defintions of rubi rules used

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3284
```
Int[1/(((a_.) + (b_.)*sin[(e_.) + (f_.)*(x_)])*Sqrt[(c_.) + (d_.)*sin[(e_.)
 + (f_.)*(x_)]]), x_Symbol] :> Simp[(2/(f*(a + b)*Sqrt[c + d]))*EllipticPi[
2*(b/(a + b)), (1/2)*(e - Pi/2 + f*x), 2*(d/(c + d))], x] /; FreeQ[{a, b, c
, d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2,
0] && GtQ[c + d, 0]
```

rule 3286
```
Int[1/(((a_.) + (b_.)*sin[(e_.) + (f_.)*(x_)])*Sqrt[(c_.) + (d_.)*sin[(e_.)
 + (f_.)*(x_)]]), x_Symbol] :> Simp[Sqrt[(c + d*Sin[e + f*x])/(c + d)]/Sqrt
[c + d*Sin[e + f*x]]   Int[1/((a + b*Sin[e + f*x])*Sqrt[c/(c + d) + (d/(c +
 d))*Sin[e + f*x]]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*
d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0] &&  !GtQ[c + d, 0]
```

rule 3420
```
Int[1/(sin[(e_.) + (f_.)*(x_)]*Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]*(
(c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[1/c   Int[1/(Sin[
e + f*x]*Sqrt[a + b*Sin[e + f*x]]), x], x] - Simp[d/c   Int[1/(Sqrt[a + b*S
in[e + f*x]]*(c + d*Sin[e + f*x])), x], x] /; FreeQ[{a, b, c, d, e, f}, x]
&& NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0]
```

3.41.4 Maple [A] (verified)

Time = 1.15 (sec) , antiderivative size = 254, normalized size of antiderivative = 1.74

method	result
default	$-\frac{2(c-d)\sqrt{\frac{c+d\sin(fx+e)}{c-d}}\,\sqrt{-\frac{(\sin(fx+e)-1)d}{c+d}}\,\sqrt{-\frac{d(1+\sin(fx+e))}{c-d}}\left(\Pi\left(\sqrt{\frac{c+d\sin(fx+e)}{c-d}},\frac{c-d}{c},\sqrt{\frac{c-d}{c+d}}\right)ad-\Pi\left(\sqrt{\frac{c+d\sin(fx+e)}{c-d}},\frac{c-d}{c},\sqrt{\right.\right.}{ac(ad-bc)\cos(fx+e)\sqrt{c+d\sin(fx+e)}\,f}$

input
```
int(1/sin(f*x+e)/(a+b*sin(f*x+e))/(c+d*sin(f*x+e))^(1/2),x,method=_RETURNV
ERBOSE)
```

output
```
-2*(c-d)*((c+d*sin(f*x+e))/(c-d))^(1/2)*(-(sin(f*x+e)-1)*d/(c+d))^(1/2)*(-
d*(1+sin(f*x+e))/(c-d))^(1/2)*(EllipticPi(((c+d*sin(f*x+e))/(c-d))^(1/2),(
c-d)/c,((c-d)/(c+d))^(1/2))*a*d-EllipticPi(((c+d*sin(f*x+e))/(c-d))^(1/2),
(c-d)/c,((c-d)/(c+d))^(1/2))*b*c+b*EllipticPi(((c+d*sin(f*x+e))/(c-d))^(1/
2),-(c-d)*b/(a*d-b*c),((c-d)/(c+d))^(1/2))*c)/a/c/(a*d-b*c)/cos(f*x+e)/(c+
d*sin(f*x+e))^(1/2)/f
```

3.41.5 Fricas [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}\,dx = \text{Timed out}$$

input
```
integrate(1/sin(f*x+e)/(a+b*sin(f*x+e))/(c+d*sin(f*x+e))^(1/2),x, algorith
m="fricas")
```

output
```
Timed out
```

3.41.6 Sympy [F]

$$\int \frac{\csc(e+fx)}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}\,dx = \int \frac{1}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}\sin(e+fx)}\,dx$$

input
```
integrate(1/sin(f*x+e)/(a+b*sin(f*x+e))/(c+d*sin(f*x+e))**(1/2),x)
```

output
```
Integral(1/((a + b*sin(e + f*x))*sqrt(c + d*sin(e + f*x))*sin(e + f*x)), x
)
```

3.41.7 Maxima [F]

$$\int \frac{\csc(e+fx)}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}\,dx = \int \frac{1}{(b\sin(fx+e)+a)\sqrt{d\sin(fx+e)+c}\sin(fx+e)}\,dx$$

input
```
integrate(1/sin(f*x+e)/(a+b*sin(f*x+e))/(c+d*sin(f*x+e))^(1/2),x, algorith
m="maxima")
```

output
```
integrate(1/((b*sin(f*x + e) + a)*sqrt(d*sin(f*x + e) + c)*sin(f*x + e)),
x)
```

3.41.8 Giac [F]

$$\int \frac{\csc(e+fx)}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}\,dx = \int \frac{1}{(b\sin(fx+e)+a)\sqrt{d\sin(fx+e)+c}\sin(fx+e)}\,dx$$

input
```
integrate(1/sin(f*x+e)/(a+b*sin(f*x+e))/(c+d*sin(f*x+e))^(1/2),x, algorith
m="giac")
```

output
```
integrate(1/((b*sin(f*x + e) + a)*sqrt(d*sin(f*x + e) + c)*sin(f*x + e)),
x)
```

3.41.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}\,dx = \int \frac{1}{\sin(e+f\,x)\,\left(a+b\,\sin(e+f\,x)\right)\,\sqrt{c+d\,\sin(e+f\,x)}}\,dx$$

input
```
int(1/(sin(e + f*x)*(a + b*sin(e + f*x))*(c + d*sin(e + f*x))^(1/2)),x)
```

output
```
int(1/(sin(e + f*x)*(a + b*sin(e + f*x))*(c + d*sin(e + f*x))^(1/2)), x)
```

3.41. $\int \frac{\csc(e+fx)}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}\,dx$

3.42 $\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}{c+d\sin(e+fx)}\,dx$

3.42.1 Optimal result

Integrand size = 39, antiderivative size = 254

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}{c+d\sin(e+fx)}\,dx = \frac{2\sqrt{a+b}\sqrt{g}\sqrt{\frac{a(1-\csc(e+fx))}{a+b}}\sqrt{\frac{a(1+\csc(e+fx))}{a-b}}\,\text{EllipticPi}\left(\frac{a+b}{b},\arcsin\left(\frac{\sqrt{g}\sqrt{a+b\sin(e+fx)}}{\sqrt{a+b}\sqrt{g\sin(e+fx)}}\right),-\frac{a+b}{a-b}\right)\tan(e+fx)}{df} - \frac{2(bc-ad)\sqrt{-\cot^2(e+fx)}\sqrt{\frac{b+a\csc(e+fx)}{a+b}}\,\text{EllipticPi}\left(\frac{2c}{c+d},\arcsin\left(\frac{\sqrt{1-\csc(e+fx)}}{\sqrt{2}}\right),\frac{2a}{a+b}\right)\sqrt{g\sin(e+fx)}\,\text{t}}{d(c+d)f\sqrt{a+b\sin(e+fx)}}$$

output
```
2*EllipticPi(g^(1/2)*(a+b*sin(f*x+e))^(1/2)/(a+b)^(1/2)/(g*sin(f*x+e))^(1/
2),(a+b)/b,((-a-b)/(a-b))^(1/2))*(a+b)^(1/2)*g^(1/2)*(a*(1-csc(f*x+e))/(a+
b))^(1/2)*(a*(1+csc(f*x+e))/(a-b))^(1/2)*tan(f*x+e)/d/f-2*(-a*d+b*c)*Ellip
ticPi(1/2*(1-csc(f*x+e))^(1/2)*2^(1/2),2*c/(c+d),2^(1/2)*(a/(a+b))^(1/2))*
(-cot(f*x+e)^2)^(1/2)*((b+a*csc(f*x+e))/(a+b))^(1/2)*(g*sin(f*x+e))^(1/2)*
tan(f*x+e)/d/(c+d)/f/(a+b*sin(f*x+e))^(1/2)
```

3.42.2 Mathematica [C] (warning: unable to verify)

Result contains complex when optimal does not.

Time = 31.90 (sec) , antiderivative size = 23019, normalized size of antiderivative = 90.63

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}{c+d\sin(e+fx)}\,dx = \text{Result too large to show}$$

input
```
Integrate[(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]])/(c + d*Sin[e + f
*x]),x]
```

output
```
Result too large to show
```

3.42.3 Rubi [A] (verified)

Time = 0.98 (sec) , antiderivative size = 254, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.128$, Rules used = {3042, 3408, 3042, 3288, 3416}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}{c+d\sin(e+fx)}\,dx$$

$$\Big\downarrow \ 3042$$

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}{c+d\sin(e+fx)}dx$$

$$\Big\downarrow \ 3408$$

$$\frac{b\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}}dx}{d}-\frac{(bc-ad)\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}dx}{d}$$

$$\Big\downarrow \ 3042$$

$$\frac{b\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}}dx}{d}-\frac{(bc-ad)\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}dx}{d}$$

$$\Big\downarrow \ 3288$$

$$\frac{2\sqrt{g}\sqrt{a+b}\tan(e+fx)\sqrt{\frac{a(1-\csc(e+fx))}{a+b}}\sqrt{\frac{a(\csc(e+fx)+1)}{a-b}}\,\text{EllipticPi}\left(\frac{a+b}{b},\arcsin\left(\frac{\sqrt{g}\sqrt{a+b\sin(e+fx)}}{\sqrt{a+b}\sqrt{g\sin(e+fx)}}\right),-\frac{a+b}{a-b}\right)}{df}-\frac{(bc-ad)\int\frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}dx}{d}$$

$\Big\downarrow$ 3416

$$\frac{2\sqrt{g}\sqrt{a+b}\tan(e+fx)\sqrt{\frac{a(1-\csc(e+fx))}{a+b}}\sqrt{\frac{a(\csc(e+fx)+1)}{a-b}}\,\text{EllipticPi}\left(\frac{a+b}{b},\arcsin\left(\frac{\sqrt{g}\sqrt{a+b\sin(e+fx)}}{\sqrt{a+b}\sqrt{g\sin(e+fx)}}\right),-\frac{a+b}{a-b}\right)}{df}-\frac{2(bc-ad)\tan(e+fx)\sqrt{-\cot^2(e+fx)}\sqrt{g\sin(e+fx)}\sqrt{\frac{a\csc(e+fx)+b}{a+b}}\,\text{EllipticPi}\left(\frac{2c}{c+d},\arcsin\left(\frac{\sqrt{1-\csc(e+fx)}}{\sqrt{2}}\right),\frac{2a}{a+b}\right)}{df(c+d)\sqrt{a+b\sin(e+fx)}}$$

input
```
Int[(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]])/(c + d*Sin[e + f*x]),x
]
```

output
```
(2*Sqrt[a + b]*Sqrt[g]*Sqrt[(a*(1 - Csc[e + f*x]))/(a + b)]*Sqrt[(a*(1 + C
sc[e + f*x]))/(a - b)]*EllipticPi[(a + b)/b, ArcSin[(Sqrt[g]*Sqrt[a + b*Si
n[e + f*x]])/(Sqrt[a + b]*Sqrt[g*Sin[e + f*x]])], -((a + b)/(a - b))]*Tan[
e + f*x])/(d*f) - (2*(b*c - a*d)*Sqrt[-Cot[e + f*x]^2]*Sqrt[(b + a*Csc[e +
 f*x])/(a + b)]*EllipticPi[(2*c)/(c + d), ArcSin[Sqrt[1 - Csc[e + f*x]]/Sq
rt[2]], (2*a)/(a + b)]*Sqrt[g*Sin[e + f*x]]*Tan[e + f*x])/(d*(c + d)*f*Sqr
t[a + b*Sin[e + f*x]])
```

3.42.3.1 Defintions of rubi rules used

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3288
```
Int[Sqrt[(b_.)*sin[(e_.) + (f_.)*(x_)]]/Sqrt[(c_) + (d_.)*sin[(e_.) + (f_.)
*(x_)]], x_Symbol] :> Simp[2*b*(Tan[e + f*x]/(d*f))*Rt[(c + d)/b, 2]*Sqrt[c
*((1 + Csc[e + f*x])/(c - d))]*Sqrt[c*((1 - Csc[e + f*x])/(c + d))]*Ellipti
cPi[(c + d)/d, ArcSin[Sqrt[c + d*Sin[e + f*x]]/Sqrt[b*Sin[e + f*x]]/Rt[(c +
 d)/b, 2]], -(c + d)/(c - d)], x] /; FreeQ[{b, c, d, e, f}, x] && NeQ[c^2 -
 d^2, 0] && PosQ[(c + d)/b]
```

rule 3408
```
Int[(Sqrt[(g_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.
)*(x_)]])/((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] :> Simp[b/d   I
nt[Sqrt[g*Sin[e + f*x]]/Sqrt[a + b*Sin[e + f*x]], x], x] - Simp[(b*c - a*d)
/d   Int[Sqrt[g*Sin[e + f*x]]/(Sqrt[a + b*Sin[e + f*x]]*(c + d*Sin[e + f*x]
)), x], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && NeQ[a
^2 - b^2, 0] && NeQ[c^2 - d^2, 0]
```

rule 3416
```
Int[Sqrt[(g_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.
)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[2*Sqrt[
-Cot[e + f*x]^2]*(Sqrt[g*Sin[e + f*x]]/(f*(c + d)*Cot[e + f*x]*Sqrt[a + b*S
in[e + f*x]]))*Sqrt[(b + a*Csc[e + f*x])/(a + b)]*EllipticPi[2*(c/(c + d)),
 ArcSin[Sqrt[1 - Csc[e + f*x]]/Sqrt[2]], 2*(a/(a + b))], x] /; FreeQ[{a, b,
 c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 -
d^2, 0]
```

3.42.4 Maple [C] (warning: unable to verify)

Result contains complex when optimal does not.

Time = 2.66 (sec) , antiderivative size = 5478, normalized size of antiderivative = 21.57

method	result	size
default	Expression too large to display	5478

input
```
int((g*sin(f*x+e))^(1/2)*(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,method=
_RETURNVERBOSE)
```

output
```
result too large to display
```

3.42.5 Fricas [F(-1)]

Timed out.

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}{c+d\sin(e+fx)}\,dx = \text{Timed out}$$

input
```
integrate((g*sin(f*x+e))^(1/2)*(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,
algorithm="fricas")
```

output
```
Timed out
```

3.42.6 Sympy [F]

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}{c+d\sin(e+fx)}\,dx = \int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}{c+d\sin(e+fx)}\,dx$$

input
```
integrate((g*sin(f*x+e))**(1/2)*(a+b*sin(f*x+e))**(1/2)/(c+d*sin(f*x+e)),x
)
```

output
```
Integral(sqrt(g*sin(e + f*x))*sqrt(a + b*sin(e + f*x))/(c + d*sin(e + f*x)
), x)
```

3.42.7 Maxima [F]

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}{c+d\sin(e+fx)}\,dx = \int \frac{\sqrt{b\sin(fx+e)+a}\sqrt{g\sin(fx+e)}}{d\sin(fx+e)+c}\,dx$$

input
```
integrate((g*sin(f*x+e))^(1/2)*(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,
algorithm="maxima")
```

output
```
integrate(sqrt(b*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))/(d*sin(f*x + e) +
c), x)
```

3.42.8 Giac [F]

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}{c+d\sin(e+fx)}\,dx = \int \frac{\sqrt{b\sin(fx+e)+a}\sqrt{g\sin(fx+e)}}{d\sin(fx+e)+c}\,dx$$

input
```
integrate((g*sin(f*x+e))^(1/2)*(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,
algorithm="giac")
```

output
```
integrate(sqrt(b*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))/(d*sin(f*x + e) +
c), x)
```

3.42.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}{c+d\sin(e+fx)}\,dx = \int \frac{\sqrt{g\,\sin\left(e+f\,x\right)}\,\sqrt{a+b\,\sin\left(e+f\,x\right)}}{c+d\,\sin\left(e+f\,x\right)}\,dx$$

input
```
int(((g*sin(e + f*x))^(1/2)*(a + b*sin(e + f*x))^(1/2))/(c + d*sin(e + f*x
)),x)
```

output
```
int(((g*sin(e + f*x))^(1/2)*(a + b*sin(e + f*x))^(1/2))/(c + d*sin(e + f*x
)), x)
```

3.43 $\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}\,dx$

3.43.1 Optimal result

Integrand size = 39, antiderivative size = 250

$$\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}\,dx = -\frac{2\sqrt{a+b}\sqrt{\frac{a(1-\csc(e+fx))}{a+b}}\sqrt{\frac{a(1+\csc(e+fx))}{a-b}}\,\text{EllipticF}\left(\arcsin\left(\frac{\sqrt{g}\sqrt{a+b\sin(e+fx)}}{\sqrt{a+b}\sqrt{g\sin(e+fx)}}\right),-\frac{a+b}{a-b}\right)\tan(e+fx)}{cf\sqrt{g}} + \frac{2(bc-ad)\sqrt{-\cot^2(e+fx)}\sqrt{\frac{b+a\csc(e+fx)}{a+b}}\,\text{EllipticPi}\left(\frac{2c}{c+d},\arcsin\left(\frac{\sqrt{1-\csc(e+fx)}}{\sqrt{2}}\right),\frac{2a}{a+b}\right)\sqrt{g\sin(e+fx)}\,\text{t}}{c(c+d)fg\sqrt{a+b\sin(e+fx)}}$$

output
```
-2*EllipticF(g^(1/2)*(a+b*sin(f*x+e))^(1/2)/(a+b)^(1/2)/(g*sin(f*x+e))^(1/
2),((-a-b)/(a-b))^(1/2))*(a+b)^(1/2)*(a*(1-csc(f*x+e))/(a+b))^(1/2)*(a*(1+
csc(f*x+e))/(a-b))^(1/2)*tan(f*x+e)/c/f/g^(1/2)+2*(-a*d+b*c)*EllipticPi(1/
2*(1-csc(f*x+e))^(1/2)*2^(1/2),2*c/(c+d),2^(1/2)*(a/(a+b))^(1/2))*(-cot(f*
x+e)^2)^(1/2)*((b+a*csc(f*x+e))/(a+b))^(1/2)*(g*sin(f*x+e))^(1/2)*tan(f*x+
e)/c/(c+d)/f/g/(a+b*sin(f*x+e))^(1/2)
```

3.43.2 Mathematica [F]

$$\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}\,dx=\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}\,dx$$

input
```
Integrate[Sqrt[a + b*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c + d*Sin[e + f*
x])),x]
```

output
```
Integrate[Sqrt[a + b*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c + d*Sin[e + f*
x])), x]
```

3.43.3 Rubi [A] (verified)

Time = 0.97 (sec) , antiderivative size = 250, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.128, Rules used = {3042, 3412, 3042, 3295, 3416}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}dx$$

$$\Big\downarrow \text{3412}$$

$$\frac{(bc-ad)\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}dx}{cg}+\frac{a\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}dx}{c}$$

$$\Big\downarrow \text{3042}$$

$$\frac{(bc-ad)\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}dx}{cg}+\frac{a\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}dx}{c}$$

$$\Big\downarrow \text{3295}$$

$$\frac{(bc-ad)\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}dx}{cg} - \frac{2\sqrt{a+b}\tan(e+fx)\sqrt{\frac{a(1-\csc(e+fx))}{a+b}}\sqrt{\frac{a(\csc(e+fx)+1)}{a-b}}\text{EllipticF}\left(\arcsin\left(\frac{\sqrt{g}\sqrt{a+b\sin(e+fx)}}{\sqrt{a+b}\sqrt{g\sin(e+fx)}}\right),-\frac{a+b}{a-b}\right)}{cf\sqrt{g}}$$

$$\Big\downarrow \text{3416}$$

$$\frac{2(bc-ad)\tan(e+fx)\sqrt{-\cot^2(e+fx)}\sqrt{g\sin(e+fx)}\sqrt{\frac{a\csc(e+fx)+b}{a+b}}\text{EllipticPi}\left(\frac{2c}{c+d},\arcsin\left(\frac{\sqrt{1-\csc(e+fx)}}{\sqrt{2}}\right),\frac{2a}{a+b}\right)}{cfg(c+d)\sqrt{a+b\sin(e+fx)}} - \frac{2\sqrt{a+b}\tan(e+fx)\sqrt{\frac{a(1-\csc(e+fx))}{a+b}}\sqrt{\frac{a(\csc(e+fx)+1)}{a-b}}\text{EllipticF}\left(\arcsin\left(\frac{\sqrt{g}\sqrt{a+b\sin(e+fx)}}{\sqrt{a+b}\sqrt{g\sin(e+fx)}}\right),-\frac{a+b}{a-b}\right)}{cf\sqrt{g}}$$

input
```
Int[Sqrt[a + b*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c + d*Sin[e + f*x])),x
]
```

output
```
(-2*Sqrt[a + b]*Sqrt[(a*(1 - Csc[e + f*x]))/(a + b)]*Sqrt[(a*(1 + Csc[e +
f*x]))/(a - b)]*EllipticF[ArcSin[(Sqrt[g]*Sqrt[a + b*Sin[e + f*x]])/(Sqrt[
a + b]*Sqrt[g*Sin[e + f*x]])], -((a + b)/(a - b))]*Tan[e + f*x])/(c*f*Sqrt
[g]) + (2*(b*c - a*d)*Sqrt[-Cot[e + f*x]^2]*Sqrt[(b + a*Csc[e + f*x])/(a +
 b)]*EllipticPi[(2*c)/(c + d), ArcSin[Sqrt[1 - Csc[e + f*x]]/Sqrt[2]], (2*
a)/(a + b)]*Sqrt[g*Sin[e + f*x]]*Tan[e + f*x])/(c*(c + d)*f*g*Sqrt[a + b*S
in[e + f*x]])
```

3.43.3.1 Defintions of rubi rules used

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3295
```
Int[1/(Sqrt[(d_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(a_) + (b_.)*sin[(e_.) + (f
_.)*(x_)]]), x_Symbol] :> Simp[-2*(Tan[e + f*x]/(a*f))*Rt[(a + b)/d, 2]*Sqr
t[a*((1 - Csc[e + f*x])/(a + b))]*Sqrt[a*((1 + Csc[e + f*x])/(a - b))]*Elli
pticF[ArcSin[Sqrt[a + b*Sin[e + f*x]]/Sqrt[d*Sin[e + f*x]]/Rt[(a + b)/d, 2]
], -(a + b)/(a - b)], x] /; FreeQ[{a, b, d, e, f}, x] && NeQ[a^2 - b^2, 0]
&& PosQ[(a + b)/d]
```

rule 3412
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(g_.)*sin[(e_.) + (f_.
)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[a/c   I
nt[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]), x], x] + Simp[(b*c -
a*d)/(c*g)   Int[Sqrt[g*Sin[e + f*x]]/(Sqrt[a + b*Sin[e + f*x]]*(c + d*Sin[
e + f*x])), x], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d, 0]
&& NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]
```

rule 3416
```
Int[Sqrt[(g_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.
)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[2*Sqrt[
-Cot[e + f*x]^2]*(Sqrt[g*Sin[e + f*x]]/(f*(c + d)*Cot[e + f*x]*Sqrt[a + b*S
in[e + f*x]]))*Sqrt[(b + a*Csc[e + f*x])/(a + b)]*EllipticPi[2*(c/(c + d)),
 ArcSin[Sqrt[1 - Csc[e + f*x]]/Sqrt[2]], 2*(a/(a + b))], x] /; FreeQ[{a, b,
 c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 -
d^2, 0]
```

3.43.4 Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 2883 vs. $2(231) = 462$.

Time = 2.50 (sec) , antiderivative size = 2884, normalized size of antiderivative = 11.54

method	result	size
default	Expression too large to display	2884

input
```
int((a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))/(g*sin(f*x+e))^(1/2),x,method=
_RETURNVERBOSE)
```

output
```
1/f*(a*d-b*c)*(2*(-a^2+b^2)^(1/2)*(-c^2+d^2)^(1/2)*EllipticPi((1/(b+(-a^2+
b^2)^(1/2))*(a*csc(f*x+e)-a*cot(f*x+e)+(-a^2+b^2)^(1/2)+b))^(1/2),(b+(-a^2
+b^2)^(1/2))*c/(c*(-a^2+b^2)^(1/2)+a*(-c^2+d^2)^(1/2)-a*d+b*c),1/2*2^(1/2)
*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1/2))*b+2*(-a^2+b^2)^(1/2)*(-c^2
+d^2)^(1/2)*EllipticPi((1/(b+(-a^2+b^2)^(1/2))*(a*csc(f*x+e)-a*cot(f*x+e)+
(-a^2+b^2)^(1/2)+b))^(1/2),(b+(-a^2+b^2)^(1/2))*c/(c*(-a^2+b^2)^(1/2)-a*(-
c^2+d^2)^(1/2)-a*d+b*c),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2)
)^(1/2))*b-4*(-a^2+b^2)^(1/2)*(-c^2+d^2)^(1/2)*EllipticF((1/(b+(-a^2+b^2)^
(1/2))*(a*csc(f*x+e)-a*cot(f*x+e)+(-a^2+b^2)^(1/2)+b))^(1/2),1/2*2^(1/2)*(
(b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1/2))*b+(-a^2+b^2)^(1/2)*EllipticP
i((1/(b+(-a^2+b^2)^(1/2))*(a*csc(f*x+e)-a*cot(f*x+e)+(-a^2+b^2)^(1/2)+b))^
(1/2),(b+(-a^2+b^2)^(1/2))*c/(c*(-a^2+b^2)^(1/2)+a*(-c^2+d^2)^(1/2)-a*d+b*
c),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1/2))*a*c-2*(-a^2+
b^2)^(1/2)*EllipticPi((1/(b+(-a^2+b^2)^(1/2))*(a*csc(f*x+e)-a*cot(f*x+e)+(
-a^2+b^2)^(1/2)+b))^(1/2),(b+(-a^2+b^2)^(1/2))*c/(c*(-a^2+b^2)^(1/2)+a*(-c
^2+d^2)^(1/2)-a*d+b*c),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))
^(1/2))*b*d-(-a^2+b^2)^(1/2)*EllipticPi((1/(b+(-a^2+b^2)^(1/2))*(a*csc(f*x
+e)-a*cot(f*x+e)+(-a^2+b^2)^(1/2)+b))^(1/2),(b+(-a^2+b^2)^(1/2))*c/(c*(-a^
2+b^2)^(1/2)-a*(-c^2+d^2)^(1/2)-a*d+b*c),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))
/(-a^2+b^2)^(1/2))^(1/2))*a*c+2*(-a^2+b^2)^(1/2)*EllipticPi((1/(b+(-a^2...
```

3.43.5 Fricas [F(-1)]

Timed out.

$$\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \text{Timed out}$$

input
```
integrate((a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))/(g*sin(f*x+e))^(1/2),x,
algorithm="fricas")
```

output
```
Timed out
```

3.43.6 Sympy [F]

$$\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \int \frac{\sqrt{a+b\sin{(e+fx)}}}{\sqrt{g\sin{(e+fx)}}\,(c+d\sin{(e+fx)})}\,dx$$

input
```
integrate((a+b*sin(f*x+e))**(1/2)/(c+d*sin(f*x+e))/(g*sin(f*x+e))**(1/2),x
)
```

output
```
Integral(sqrt(a + b*sin(e + f*x))/(sqrt(g*sin(e + f*x))*(c + d*sin(e + f*x
))), x)
```

3.43.7 Maxima [F]

$$\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \int \frac{\sqrt{b\sin{(fx+e)}+a}}{(d\sin{(fx+e)}+c)\sqrt{g\sin{(fx+e)}}}\,dx$$

input
```
integrate((a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))/(g*sin(f*x+e))^(1/2),x,
algorithm="maxima")
```

output
```
integrate(sqrt(b*sin(f*x + e) + a)/((d*sin(f*x + e) + c)*sqrt(g*sin(f*x +
e))), x)
```

3.43.8 Giac [F]

$$\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \int \frac{\sqrt{b\sin{(fx+e)}+a}}{(d\sin{(fx+e)}+c)\sqrt{g\sin{(fx+e)}}}\,dx$$

input
```
integrate((a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))/(g*sin(f*x+e))^(1/2),x,
algorithm="giac")
```

output
```
integrate(sqrt(b*sin(f*x + e) + a)/((d*sin(f*x + e) + c)*sqrt(g*sin(f*x +
e))), x)
```

3.43.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{g\sin(e+fx)}(c+d\sin(e+fx))}\,dx=\int \frac{\sqrt{a+b\,\sin\left(e+f\,x\right)}}{\sqrt{g\,\sin\left(e+f\,x\right)}\,\left(c+d\,\sin\left(e+f\,x\right)\right)}\,dx$$

input
```
int((a + b*sin(e + f*x))^(1/2)/((g*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x)
)),x)
```

output
```
int((a + b*sin(e + f*x))^(1/2)/((g*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x)
)), x)
```

3.44 $\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}\,dx$

3.44.1 Optimal result

Integrand size = 39, antiderivative size = 114

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \frac{2\sqrt{-\cot^2(e+fx)}\sqrt{\frac{b+a\csc(e+fx)}{a+b}}\,\text{EllipticPi}\left(\frac{2c}{c+d},\arcsin\left(\frac{\sqrt{1-\csc(e+fx)}}{\sqrt{2}}\right),\frac{2a}{a+b}\right)\sqrt{g\sin(e+fx)}\tan(e+fx)}{(c+d)f\sqrt{a+b\sin(e+fx)}}$$

output
```
2*EllipticPi(1/2*(1-csc(f*x+e))^(1/2)*2^(1/2),2*c/(c+d),2^(1/2)*(a/(a+b))^
(1/2))*(-cot(f*x+e)^2)^(1/2)*((b+a*csc(f*x+e))/(a+b))^(1/2)*(g*sin(f*x+e))
^(1/2)*tan(f*x+e)/(c+d)/f/(a+b*sin(f*x+e))^(1/2)
```

3.44.2 Mathematica [F]

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}\,dx$$

input
```
Integrate[Sqrt[g*Sin[e + f*x]]/(Sqrt[a + b*Sin[e + f*x]]*(c + d*Sin[e + f*
x])),x]
```

output
```
Integrate[Sqrt[g*Sin[e + f*x]]/(Sqrt[a + b*Sin[e + f*x]]*(c + d*Sin[e + f*
x])), x]
```

3.44.3 Rubi [A] (verified)

Time = 0.41 (sec) , antiderivative size = 114, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}} = 0.051$, Rules used = {3042, 3416}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}dx$$

$$\Big\downarrow \; 3042$$

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}dx$$

$$\Big\downarrow \; 3416$$

$$\frac{2\tan(e+fx)\sqrt{-\cot^2(e+fx)}\sqrt{g\sin(e+fx)}\sqrt{\frac{a\csc(e+fx)+b}{a+b}}\text{EllipticPi}\left(\frac{2c}{c+d},\arcsin\left(\frac{\sqrt{1-\csc(e+fx)}}{\sqrt{2}}\right),\frac{2a}{a+b}\right)}{f(c+d)\sqrt{a+b\sin(e+fx)}}$$

input
```
Int[Sqrt[g*Sin[e + f*x]]/(Sqrt[a + b*Sin[e + f*x]]*(c + d*Sin[e + f*x])),x
]
```

output
```
(2*Sqrt[-Cot[e + f*x]^2]*Sqrt[(b + a*Csc[e + f*x])/(a + b)]*EllipticPi[(2*
c)/(c + d), ArcSin[Sqrt[1 - Csc[e + f*x]]/Sqrt[2]], (2*a)/(a + b)]*Sqrt[g*
Sin[e + f*x]]*Tan[e + f*x])/((c + d)*f*Sqrt[a + b*Sin[e + f*x]])
```

3.44.3.1 Defintions of rubi rules used

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3416
```
Int[Sqrt[(g_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.
)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[2*Sqrt[
-Cot[e + f*x]^2]*(Sqrt[g*Sin[e + f*x]]/(f*(c + d)*Cot[e + f*x]*Sqrt[a + b*S
in[e + f*x]]))*Sqrt[(b + a*Csc[e + f*x])/(a + b)]*EllipticPi[2*(c/(c + d)),
 ArcSin[Sqrt[1 - Csc[e + f*x]]/Sqrt[2]], 2*(a/(a + b))], x] /; FreeQ[{a, b,
 c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 -
d^2, 0]
```

3.44.4 Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 2582 vs. $2(107) = 214$.

Time = 2.80 (sec) , antiderivative size = 2583, normalized size of antiderivative = 22.66

method	result	size
default	Expression too large to display	2583

input
```
int((g*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))/(a+b*sin(f*x+e))^(1/2),x,method=
_RETURNVERBOSE)
```

output
```
-1/f*(EllipticPi((1/(b+(-a^2+b^2)^(1/2))*(a*csc(f*x+e)-a*cot(f*x+e)+(-a^2+
b^2)^(1/2)+b))^(1/2),(b+(-a^2+b^2)^(1/2))*c/(c*(-a^2+b^2)^(1/2)+a*(-c^2+d^
2)^(1/2)-a*d+b*c),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1/2
))*a^2*d-(-c^2+d^2)^(1/2)*EllipticPi((1/(b+(-a^2+b^2)^(1/2))*(a*csc(f*x+e)
-a*cot(f*x+e)+(-a^2+b^2)^(1/2)+b))^(1/2),(b+(-a^2+b^2)^(1/2))*c/(c*(-a^2+b
^2)^(1/2)+a*(-c^2+d^2)^(1/2)-a*d+b*c),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-
a^2+b^2)^(1/2))^(1/2))*a^2+EllipticPi((1/(b+(-a^2+b^2)^(1/2))*(a*csc(f*x+e
)-a*cot(f*x+e)+(-a^2+b^2)^(1/2)+b))^(1/2),(b+(-a^2+b^2)^(1/2))*c/(c*(-a^2+
b^2)^(1/2)+a*(-c^2+d^2)^(1/2)-a*d+b*c),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(
-a^2+b^2)^(1/2))^(1/2))*a*b*c+(-a^2+b^2)^(1/2)*EllipticPi((1/(b+(-a^2+b^2)
^(1/2))*(a*csc(f*x+e)-a*cot(f*x+e)+(-a^2+b^2)^(1/2)+b))^(1/2),(b+(-a^2+b^2
)^(1/2))*c/(c*(-a^2+b^2)^(1/2)+a*(-c^2+d^2)^(1/2)-a*d+b*c),1/2*2^(1/2)*((b
+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1/2))*a*c-2*EllipticPi((1/(b+(-a^2+b
^2)^(1/2))*(a*csc(f*x+e)-a*cot(f*x+e)+(-a^2+b^2)^(1/2)+b))^(1/2),(b+(-a^2+
b^2)^(1/2))*c/(c*(-a^2+b^2)^(1/2)+a*(-c^2+d^2)^(1/2)-a*d+b*c),1/2*2^(1/2)*
((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1/2))*b^2*d+2*(-c^2+d^2)^(1/2)*El
lipticPi((1/(b+(-a^2+b^2)^(1/2))*(a*csc(f*x+e)-a*cot(f*x+e)+(-a^2+b^2)^(1/
2)+b))^(1/2),(b+(-a^2+b^2)^(1/2))*c/(c*(-a^2+b^2)^(1/2)+a*(-c^2+d^2)^(1/2)
-a*d+b*c),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1/2))*b^2-2
*(-a^2+b^2)^(1/2)*EllipticPi((1/(b+(-a^2+b^2)^(1/2))*(a*csc(f*x+e)-a*co...
```

3.44.5 Fricas [F(-1)]

Timed out.

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \text{Timed out}$$

input
```
integrate((g*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))/(a+b*sin(f*x+e))^(1/2),x,
algorithm="fricas")
```

output
```
Timed out
```

3.44.6 Sympy [F]

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \int \frac{\sqrt{g\sin{(e+fx)}}}{\sqrt{a+b\sin{(e+fx)}}\,(c+d\sin{(e+fx)})}\,dx$$

input
```
integrate((g*sin(f*x+e))**(1/2)/(c+d*sin(f*x+e))/(a+b*sin(f*x+e))**(1/2),x
)
```

output
```
Integral(sqrt(g*sin(e + f*x))/(sqrt(a + b*sin(e + f*x))*(c + d*sin(e + f*x
))), x)
```

3.44.7 Maxima [F]

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \int \frac{\sqrt{g\sin{(fx+e)}}}{\sqrt{b\sin{(fx+e)}+a}(d\sin{(fx+e)}+c)}\,dx$$

input
```
integrate((g*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))/(a+b*sin(f*x+e))^(1/2),x,
algorithm="maxima")
```

output
```
integrate(sqrt(g*sin(f*x + e))/(sqrt(b*sin(f*x + e) + a)*(d*sin(f*x + e) +
 c)), x)
```

3.44.8 Giac [F]

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \int \frac{\sqrt{g\sin{(fx+e)}}}{\sqrt{b\sin{(fx+e)}+a}(d\sin{(fx+e)}+c)}\,dx$$

input
```
integrate((g*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))/(a+b*sin(f*x+e))^(1/2),x,
algorithm="giac")
```

output
```
integrate(sqrt(g*sin(f*x + e))/(sqrt(b*sin(f*x + e) + a)*(d*sin(f*x + e) +
 c)), x)
```

3.44.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}\,dx$$
$$=\int \frac{\sqrt{g\,\sin\left(e+f\,x\right)}}{\sqrt{a+b\,\sin\left(e+f\,x\right)}\,\left(c+d\,\sin\left(e+f\,x\right)\right)}\,dx$$

input
```
int((g*sin(e + f*x))^(1/2)/((a + b*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x)
)),x)
```

output
```
int((g*sin(e + f*x))^(1/2)/((a + b*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x)
)), x)
```

3.45 $\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}\,dx$

3.45.1 Optimal result

Integrand size = 39, antiderivative size = 246

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}\,dx = -\frac{2\sqrt{a+b}\sqrt{\frac{a(1-\csc(e+fx))}{a+b}}\sqrt{\frac{a(1+\csc(e+fx))}{a-b}}\,\text{EllipticF}\left(\arcsin\left(\frac{\sqrt{g}\sqrt{a+b\sin(e+fx)}}{\sqrt{a+b}\sqrt{g\sin(e+fx)}}\right),-\frac{a+b}{a-b}\right)\tan(e+fx)}{acf\sqrt{g}} - \frac{2d\sqrt{-\cot^2(e+fx)}\sqrt{\frac{b+a\csc(e+fx)}{a+b}}\,\text{EllipticPi}\left(\frac{2c}{c+d},\arcsin\left(\frac{\sqrt{1-\csc(e+fx)}}{\sqrt{2}}\right),\frac{2a}{a+b}\right)\sqrt{g\sin(e+fx)}\tan(e+fx)}{c(c+d)fg\sqrt{a+b\sin(e+fx)}}$$

output
```
-2*EllipticF(g^(1/2)*(a+b*sin(f*x+e))^(1/2)/(a+b)^(1/2)/(g*sin(f*x+e))^(1/
2),((-a-b)/(a-b))^(1/2))*(a+b)^(1/2)*(a*(1-csc(f*x+e))/(a+b))^(1/2)*(a*(1+
csc(f*x+e))/(a-b))^(1/2)*tan(f*x+e)/a/c/f/g^(1/2)-2*d*EllipticPi(1/2*(1-cs
c(f*x+e))^(1/2)*2^(1/2),2*c/(c+d),2^(1/2)*(a/(a+b))^(1/2))*(-cot(f*x+e)^2)
^(1/2)*((b+a*csc(f*x+e))/(a+b))^(1/2)*(g*sin(f*x+e))^(1/2)*tan(f*x+e)/c/(c
+d)/f/g/(a+b*sin(f*x+e))^(1/2)
```

3.45.2 Mathematica [B] (warning: unable to verify)

Leaf count is larger than twice the leaf count of optimal. 5612 vs. $2(246) = 492$.

Time = 31.09 (sec) , antiderivative size = 5612, normalized size of antiderivative = 22.81

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \text{Result too large to show}$$

input
```
Integrate[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]*(c + d*Sin[e +
f*x])),x]
```

output
```
Result too large to show
```

3.45.3 Rubi [A] (verified)

Time = 0.99 (sec) , antiderivative size = 246, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.128$, Rules used = {3042, 3418, 3042, 3295, 3416}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}dx$$

$$\Big\downarrow \text{3418}$$

$$\frac{\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}dx}{c}-\frac{d\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}dx}{cg}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}}dx}{c}-\frac{d\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}dx}{cg}$$

$$\Big\downarrow \text{3295}$$

$$-\frac{d\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}dx}{cg}-\frac{2\sqrt{a+b}\tan(e+fx)\sqrt{\frac{a(1-\csc(e+fx))}{a+b}}\sqrt{\frac{a(\csc(e+fx)+1)}{a-b}}\text{EllipticF}\left(\arcsin\left(\frac{\sqrt{g}\sqrt{a+b\sin(e+fx)}}{\sqrt{a+b}\sqrt{g\sin(e+fx)}}\right),-\frac{a+b}{a-b}\right)}{acf\sqrt{g}}$$

$$\Big\downarrow \text{ 3416}$$

$$-\frac{2d\tan(e+fx)\sqrt{-\cot^2(e+fx)}\sqrt{g\sin(e+fx)}\sqrt{\frac{a\csc(e+fx)+b}{a+b}}\text{EllipticPi}\left(\frac{2c}{c+d},\arcsin\left(\frac{\sqrt{1-\csc(e+fx)}}{\sqrt{2}}\right),\frac{2a}{a+b}\right)}{cfg(c+d)\sqrt{a+b\sin(e+fx)}}-\frac{2\sqrt{a+b}\tan(e+fx)\sqrt{\frac{a(1-\csc(e+fx))}{a+b}}\sqrt{\frac{a(\csc(e+fx)+1)}{a-b}}\text{EllipticF}\left(\arcsin\left(\frac{\sqrt{g}\sqrt{a+b\sin(e+fx)}}{\sqrt{a+b}\sqrt{g\sin(e+fx)}}\right),-\frac{a+b}{a-b}\right)}{acf\sqrt{g}}$$

input
```
Int[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]*(c + d*Sin[e + f*x]))
,x]
```

output
```
(-2*Sqrt[a + b]*Sqrt[(a*(1 - Csc[e + f*x]))/(a + b)]*Sqrt[(a*(1 + Csc[e +
f*x]))/(a - b)]*EllipticF[ArcSin[(Sqrt[g]*Sqrt[a + b*Sin[e + f*x]])/(Sqrt[
a + b]*Sqrt[g*Sin[e + f*x]])], -((a + b)/(a - b))]*Tan[e + f*x])/(a*c*f*Sq
rt[g]) - (2*d*Sqrt[-Cot[e + f*x]^2]*Sqrt[(b + a*Csc[e + f*x])/(a + b)]*Ell
ipticPi[(2*c)/(c + d), ArcSin[Sqrt[1 - Csc[e + f*x]]/Sqrt[2]], (2*a)/(a +
b)]*Sqrt[g*Sin[e + f*x]]*Tan[e + f*x])/(c*(c + d)*f*g*Sqrt[a + b*Sin[e + f
*x]])
```

3.45.3.1 Defintions of rubi rules used

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3295
```
Int[1/(Sqrt[(d_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(a_) + (b_.)*sin[(e_.) + (f
_.)*(x_)]]), x_Symbol] :> Simp[-2*(Tan[e + f*x]/(a*f))*Rt[(a + b)/d, 2]*Sqr
t[a*((1 - Csc[e + f*x])/(a + b))]*Sqrt[a*((1 + Csc[e + f*x])/(a - b))]*Elli
pticF[ArcSin[Sqrt[a + b*Sin[e + f*x]]/Sqrt[d*Sin[e + f*x]]/Rt[(a + b)/d, 2]
], -(a + b)/(a - b)], x] /; FreeQ[{a, b, d, e, f}, x] && NeQ[a^2 - b^2, 0]
&& PosQ[(a + b)/d]
```

rule 3416
```
Int[Sqrt[(g_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.
)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[2*Sqrt[
-Cot[e + f*x]^2]*(Sqrt[g*Sin[e + f*x]]/(f*(c + d)*Cot[e + f*x]*Sqrt[a + b*S
in[e + f*x]]))*Sqrt[(b + a*Csc[e + f*x])/(a + b)]*EllipticPi[2*(c/(c + d)),
 ArcSin[Sqrt[1 - Csc[e + f*x]]/Sqrt[2]], 2*(a/(a + b))], x] /; FreeQ[{a, b,
 c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 -
d^2, 0]
```

rule 3418
```
Int[1/(Sqrt[(g_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(a_) + (b_.)*sin[(e_.) + (f
_.)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[1/c
 Int[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]), x], x] - Simp[d/(c*
g)   Int[Sqrt[g*Sin[e + f*x]]/(Sqrt[a + b*Sin[e + f*x]]*(c + d*Sin[e + f*x]
)), x], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && NeQ[a
^2 - b^2, 0] && NeQ[c^2 - d^2, 0]
```

3.45.4 Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 3342 vs. $2(227) = 454$.

Time = 2.86 (sec) , antiderivative size = 3343, normalized size of antiderivative = 13.59

method	result	size
default	Expression too large to display	3343

input
```
int(1/(c+d*sin(f*x+e))/(g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2),x,metho
d=_RETURNVERBOSE)
```

output
```
1/f*(2*EllipticPi((-(a*cot(f*x+e)-a*csc(f*x+e)-(-a^2+b^2)^(1/2)-b)/(b+(-a^
2+b^2)^(1/2)))^(1/2),(b+(-a^2+b^2)^(1/2))*c/(c*(-a^2+b^2)^(1/2)-a*(-c^2+d^
2)^(1/2)-a*d+b*c),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1/2
))*a*b*d*(-a^2+b^2)^(1/2)*(-c^2+d^2)^(1/2)-EllipticPi((-(a*cot(f*x+e)-a*cs
c(f*x+e)-(-a^2+b^2)^(1/2)-b)/(b+(-a^2+b^2)^(1/2)))^(1/2),(b+(-a^2+b^2)^(1/
2))*c/(c*(-a^2+b^2)^(1/2)-a*(-c^2+d^2)^(1/2)-a*d+b*c),1/2*2^(1/2)*((b+(-a^
2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1/2))*a^2*c*d*(-a^2+b^2)^(1/2)+2*Elliptic
Pi((-(a*cot(f*x+e)-a*csc(f*x+e)-(-a^2+b^2)^(1/2)-b)/(b+(-a^2+b^2)^(1/2)))^
(1/2),(b+(-a^2+b^2)^(1/2))*c/(c*(-a^2+b^2)^(1/2)-a*(-c^2+d^2)^(1/2)-a*d+b*
c),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1/2))*a*b*d^2*(-a^
2+b^2)^(1/2)+2*EllipticPi((-(a*cot(f*x+e)-a*csc(f*x+e)-(-a^2+b^2)^(1/2)-b)
/(b+(-a^2+b^2)^(1/2)))^(1/2),(b+(-a^2+b^2)^(1/2))*c/(c*(-a^2+b^2)^(1/2)+a*
(-c^2+d^2)^(1/2)-a*d+b*c),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/
2))^(1/2))*a*b*d*(-a^2+b^2)^(1/2)*(-c^2+d^2)^(1/2)+EllipticPi((-(a*cot(f*x
+e)-a*csc(f*x+e)-(-a^2+b^2)^(1/2)-b)/(b+(-a^2+b^2)^(1/2)))^(1/2),(b+(-a^2+
b^2)^(1/2))*c/(c*(-a^2+b^2)^(1/2)+a*(-c^2+d^2)^(1/2)-a*d+b*c),1/2*2^(1/2)*
((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1/2))*a^2*c*d*(-a^2+b^2)^(1/2)-2*
EllipticPi((-(a*cot(f*x+e)-a*csc(f*x+e)-(-a^2+b^2)^(1/2)-b)/(b+(-a^2+b^2)^
(1/2)))^(1/2),(b+(-a^2+b^2)^(1/2))*c/(c*(-a^2+b^2)^(1/2)+a*(-c^2+d^2)^(1/2
)-a*d+b*c),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1/2))*a...
```

3.45.5 Fricas [F]

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}\,dx = \int \frac{1}{\sqrt{b\sin\left(fx+e\right)+a}(d\sin\left(fx+e\right)+c)\sqrt{g\sin\left(fx+e\right)}}\,dx$$

input
```
integrate(1/(c+d*sin(f*x+e))/(g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2),x
, algorithm="fricas")
```

output
```
integral(-sqrt(b*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))/((b*c + a*d)*g*cos
(f*x + e)^2 - (b*c + a*d)*g + (b*d*g*cos(f*x + e)^2 - (a*c + b*d)*g)*sin(f
*x + e)), x)
```

3.45.6 Sympy [F]

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}\,dx$$
$$=\int \frac{1}{\sqrt{g\sin{(e+fx)}}\sqrt{a+b\sin{(e+fx)}}\,(c+d\sin{(e+fx)})}\,dx$$

input
```
integrate(1/(c+d*sin(f*x+e))/(g*sin(f*x+e))**(1/2)/(a+b*sin(f*x+e))**(1/2)
,x)
```

output
```
Integral(1/(sqrt(g*sin(e + f*x))*sqrt(a + b*sin(e + f*x))*(c + d*sin(e + f
*x))), x)
```

3.45.7 Maxima [F]

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}\,dx$$
$$=\int \frac{1}{\sqrt{b\sin{(fx+e)}+a}(d\sin{(fx+e)}+c)\sqrt{g\sin{(fx+e)}}}\,dx$$

input
```
integrate(1/(c+d*sin(f*x+e))/(g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2),x
, algorithm="maxima")
```

output
```
integrate(1/(sqrt(b*sin(f*x + e) + a)*(d*sin(f*x + e) + c)*sqrt(g*sin(f*x
+ e))), x)
```

3.45.8 Giac [F]

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}\,dx$$
$$=\int \frac{1}{\sqrt{b\sin{(fx+e)}+a}(d\sin{(fx+e)}+c)\sqrt{g\sin{(fx+e)}}}\,dx$$

input
```
integrate(1/(c+d*sin(f*x+e))/(g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2),x
, algorithm="giac")
```

output
```
integrate(1/(sqrt(b*sin(f*x + e) + a)*(d*sin(f*x + e) + c)*sqrt(g*sin(f*x
+ e))), x)
```

3.45.9 Mupad [F(-1)]

Timed out.

$$\int \frac{1}{\sqrt{g\sin(e+fx)}\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}\,dx$$
$$=\int \frac{1}{\sqrt{g\,\sin\left(e+f\,x\right)}\,\sqrt{a+b\,\sin\left(e+f\,x\right)}\,\left(c+d\,\sin\left(e+f\,x\right)\right)}\,dx$$

input
```
int(1/((g*sin(e + f*x))^(1/2)*(a + b*sin(e + f*x))^(1/2)*(c + d*sin(e + f*
x))),x)
```

output
```
int(1/((g*sin(e + f*x))^(1/2)*(a + b*sin(e + f*x))^(1/2)*(c + d*sin(e + f*
x))), x)
```

3.46 $\int \frac{\sqrt{g\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}{a+b\sin(e+fx)}\,dx$

3.46.1 Optimal result

Integrand size = 39, antiderivative size = 254

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}{a+b\sin(e+fx)}\,dx = \frac{2\sqrt{c+d}\sqrt{g}\sqrt{\frac{c(1-\csc(e+fx))}{c+d}}\sqrt{\frac{c(1+\csc(e+fx))}{c-d}}\,\text{EllipticPi}\left(\frac{c+d}{d},\arcsin\left(\frac{\sqrt{g}\sqrt{c+d\sin(e+fx)}}{\sqrt{c+d}\sqrt{g\sin(e+fx)}}\right),-\frac{c+d}{c-d}\right)\tan(e+fx)}{bf} + \frac{2(bc-ad)\sqrt{-\cot^2(e+fx)}\sqrt{\frac{d+c\csc(e+fx)}{c+d}}\,\text{EllipticPi}\left(\frac{2a}{a+b},\arcsin\left(\frac{\sqrt{1-\csc(e+fx)}}{\sqrt{2}}\right),\frac{2c}{c+d}\right)\sqrt{g\sin(e+fx)}\,\text{t}}{b(a+b)f\sqrt{c+d\sin(e+fx)}}$$

output
```
2*EllipticPi(g^(1/2)*(c+d*sin(f*x+e))^(1/2)/(c+d)^(1/2)/(g*sin(f*x+e))^(1/
2),(c+d)/d,((-c-d)/(c-d))^(1/2))*(c+d)^(1/2)*g^(1/2)*(c*(1-csc(f*x+e))/(c+
d))^(1/2)*(c*(1+csc(f*x+e))/(c-d))^(1/2)*tan(f*x+e)/b/f+2*(-a*d+b*c)*Ellip
ticPi(1/2*(1-csc(f*x+e))^(1/2)*2^(1/2),2*a/(a+b),2^(1/2)*(c/(c+d))^(1/2))*
(-cot(f*x+e)^2)^(1/2)*((d+c*csc(f*x+e))/(c+d))^(1/2)*(g*sin(f*x+e))^(1/2)*
tan(f*x+e)/b/(a+b)/f/(c+d*sin(f*x+e))^(1/2)
```

3.46.2 Mathematica [C] (warning: unable to verify)

Result contains complex when optimal does not.

Time = 32.30 (sec) , antiderivative size = 23019, normalized size of antiderivative = 90.63

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}{a+b\sin(e+fx)}\,dx = \text{Result too large to show}$$

input
```
Integrate[(Sqrt[g*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]])/(a + b*Sin[e + f
*x]),x]
```

output
```
Result too large to show
```

3.46.3 Rubi [A] (verified)

Time = 0.98 (sec) , antiderivative size = 254, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.128$, Rules used = {3042, 3408, 3042, 3288, 3416}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}{a+b\sin(e+fx)}\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}{a+b\sin(e+fx)}dx$$

$$\Big\downarrow \text{3408}$$

$$\frac{(bc-ad)\int \frac{\sqrt{g\sin(e+fx)}}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}dx}{b}+\frac{d\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}}dx}{b}$$

$$\Big\downarrow \text{3042}$$

$$\frac{(bc-ad)\int \frac{\sqrt{g\sin(e+fx)}}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}dx}{b}+\frac{d\int \frac{\sqrt{g\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}}dx}{b}$$

$$\Big\downarrow \text{3288}$$

$$\frac{(bc-ad)\int \frac{\sqrt{g\sin(e+fx)}}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}dx}{b}+$$
$$\frac{2\sqrt{g}\sqrt{c+d}\tan(e+fx)\sqrt{\frac{c(1-\csc(e+fx))}{c+d}}\sqrt{\frac{c(\csc(e+fx)+1)}{c-d}}\text{EllipticPi}\left(\frac{c+d}{d},\arcsin\left(\frac{\sqrt{g}\sqrt{c+d\sin(e+fx)}}{\sqrt{c+d}\sqrt{g\sin(e+fx)}}\right),-\frac{c+d}{c-d}\right)}{bf}$$

$$\Big\downarrow \; 3416$$

$$\frac{2(bc-ad)\tan(e+fx)\sqrt{-\cot^2(e+fx)}\sqrt{g\sin(e+fx)}\sqrt{\frac{c\csc(e+fx)+d}{c+d}}\text{EllipticPi}\left(\frac{2a}{a+b},\arcsin\left(\frac{\sqrt{1-\csc(e+fx)}}{\sqrt{2}}\right),\frac{2c}{c+d}\right)}{bf(a+b)\sqrt{c+d\sin(e+fx)}}$$
$$\frac{2\sqrt{g}\sqrt{c+d}\tan(e+fx)\sqrt{\frac{c(1-\csc(e+fx))}{c+d}}\sqrt{\frac{c(\csc(e+fx)+1)}{c-d}}\text{EllipticPi}\left(\frac{c+d}{d},\arcsin\left(\frac{\sqrt{g}\sqrt{c+d\sin(e+fx)}}{\sqrt{c+d}\sqrt{g\sin(e+fx)}}\right),-\frac{c+d}{c-d}\right)}{bf}$$

input
```
Int[(Sqrt[g*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]])/(a + b*Sin[e + f*x]),x
]
```

output
```
(2*Sqrt[c + d]*Sqrt[g]*Sqrt[(c*(1 - Csc[e + f*x]))/(c + d)]*Sqrt[(c*(1 + C
sc[e + f*x]))/(c - d)]*EllipticPi[(c + d)/d, ArcSin[(Sqrt[g]*Sqrt[c + d*Si
n[e + f*x]])/(Sqrt[c + d]*Sqrt[g*Sin[e + f*x]])], -((c + d)/(c - d))]*Tan[
e + f*x])/(b*f) + (2*(b*c - a*d)*Sqrt[-Cot[e + f*x]^2]*Sqrt[(d + c*Csc[e +
 f*x])/(c + d)]*EllipticPi[(2*a)/(a + b), ArcSin[Sqrt[1 - Csc[e + f*x]]/Sq
rt[2]], (2*c)/(c + d)]*Sqrt[g*Sin[e + f*x]]*Tan[e + f*x])/(b*(a + b)*f*Sqr
t[c + d*Sin[e + f*x]])
```

3.46.3.1 Defintions of rubi rules used

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3288
```
Int[Sqrt[(b_.)*sin[(e_.) + (f_.)*(x_)]]/Sqrt[(c_) + (d_.)*sin[(e_.) + (f_.)
*(x_)]], x_Symbol] :> Simp[2*b*(Tan[e + f*x]/(d*f))*Rt[(c + d)/b, 2]*Sqrt[c
*((1 + Csc[e + f*x])/(c - d))]*Sqrt[c*((1 - Csc[e + f*x])/(c + d))]*Ellipti
cPi[(c + d)/d, ArcSin[Sqrt[c + d*Sin[e + f*x]]/Sqrt[b*Sin[e + f*x]]/Rt[(c +
 d)/b, 2]], -(c + d)/(c - d)], x] /; FreeQ[{b, c, d, e, f}, x] && NeQ[c^2 -
 d^2, 0] && PosQ[(c + d)/b]
```

rule 3408
```
Int[(Sqrt[(g_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.
)*(x_)]])/((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] :> Simp[b/d   I
nt[Sqrt[g*Sin[e + f*x]]/Sqrt[a + b*Sin[e + f*x]], x], x] - Simp[(b*c - a*d)
/d   Int[Sqrt[g*Sin[e + f*x]]/(Sqrt[a + b*Sin[e + f*x]]*(c + d*Sin[e + f*x]
)), x], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && NeQ[a
^2 - b^2, 0] && NeQ[c^2 - d^2, 0]
```

rule 3416
```
Int[Sqrt[(g_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.
)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[2*Sqrt[
-Cot[e + f*x]^2]*(Sqrt[g*Sin[e + f*x]]/(f*(c + d)*Cot[e + f*x]*Sqrt[a + b*S
in[e + f*x]]))*Sqrt[(b + a*Csc[e + f*x])/(a + b)]*EllipticPi[2*(c/(c + d)),
 ArcSin[Sqrt[1 - Csc[e + f*x]]/Sqrt[2]], 2*(a/(a + b))], x] /; FreeQ[{a, b,
 c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 -
d^2, 0]
```

3.46.4 Maple [C] (warning: unable to verify)

Result contains complex when optimal does not.

Time = 2.69 (sec) , antiderivative size = 5489, normalized size of antiderivative = 21.61

method	result	size
default	Expression too large to display	5489

input
```
int((c+d*sin(f*x+e))^(1/2)*(g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e)),x,method=
_RETURNVERBOSE)
```

output
```
result too large to display
```

3.46.5 Fricas [F(-1)]

Timed out.

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}{a+b\sin(e+fx)}\,dx = \text{Timed out}$$

input
```
integrate((c+d*sin(f*x+e))^(1/2)*(g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e)),x,
algorithm="fricas")
```

output
```
Timed out
```

3.46.6 Sympy [F]

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}{a+b\sin(e+fx)}\,dx = \int \frac{\sqrt{g\sin\left(e+fx\right)}\sqrt{c+d\sin\left(e+fx\right)}}{a+b\sin\left(e+fx\right)}\,dx$$

input
```
integrate((c+d*sin(f*x+e))**(1/2)*(g*sin(f*x+e))**(1/2)/(a+b*sin(f*x+e)),x
)
```

output
```
Integral(sqrt(g*sin(e + f*x))*sqrt(c + d*sin(e + f*x))/(a + b*sin(e + f*x)
), x)
```

3.46.7 Maxima [F]

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}{a+b\sin(e+fx)}\,dx = \int \frac{\sqrt{d\sin\left(fx+e\right)+c}\sqrt{g\sin\left(fx+e\right)}}{b\sin\left(fx+e\right)+a}\,dx$$

input
```
integrate((c+d*sin(f*x+e))^(1/2)*(g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e)),x,
algorithm="maxima")
```

output
```
integrate(sqrt(d*sin(f*x + e) + c)*sqrt(g*sin(f*x + e))/(b*sin(f*x + e) +
a), x)
```

3.46.8 Giac [F]

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}{a+b\sin(e+fx)}\,dx = \int \frac{\sqrt{d\sin\left(fx+e\right)+c}\sqrt{g\sin\left(fx+e\right)}}{b\sin\left(fx+e\right)+a}\,dx$$

input
```
integrate((c+d*sin(f*x+e))^(1/2)*(g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e)),x,
algorithm="giac")
```

output
```
integrate(sqrt(d*sin(f*x + e) + c)*sqrt(g*sin(f*x + e))/(b*sin(f*x + e) +
a), x)
```

3.46.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{g\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}{a+b\sin(e+fx)}\,dx = \int \frac{\sqrt{g\,\sin\left(e+f\,x\right)}\,\sqrt{c+d\,\sin\left(e+f\,x\right)}}{a+b\,\sin\left(e+f\,x\right)}\,dx$$

input
```
int(((g*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))^(1/2))/(a + b*sin(e + f*x
)),x)
```

output
```
int(((g*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))^(1/2))/(a + b*sin(e + f*x
)), x)
```

3.47 $\int \frac{\sqrt{g\sin(e+fx)}}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}\,dx$

3.47.1 Optimal result

Integrand size = 39, antiderivative size = 114

$$\int \frac{\sqrt{g\sin(e+fx)}}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}\,dx = \frac{2\sqrt{-\cot^2(e+fx)}\sqrt{\frac{d+c\csc(e+fx)}{c+d}}\,\text{EllipticPi}\left(\frac{2a}{a+b},\arcsin\left(\frac{\sqrt{1-\csc(e+fx)}}{\sqrt{2}}\right),\frac{2c}{c+d}\right)\sqrt{g\sin(e+fx)}\tan(e+fx)}{(a+b)f\sqrt{c+d\sin(e+fx)}}$$

output
```
2*EllipticPi(1/2*(1-csc(f*x+e))^(1/2)*2^(1/2),2*a/(a+b),2^(1/2)*(c/(c+d))^
(1/2))*(-cot(f*x+e)^2)^(1/2)*((d+c*csc(f*x+e))/(c+d))^(1/2)*(g*sin(f*x+e))
^(1/2)*tan(f*x+e)/(a+b)/f/(c+d*sin(f*x+e))^(1/2)
```

3.47.2 Mathematica [F]

$$\int \frac{\sqrt{g\sin(e+fx)}}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}\,dx = \int \frac{\sqrt{g\sin(e+fx)}}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}\,dx$$

input
```
Integrate[Sqrt[g*Sin[e + f*x]]/((a + b*Sin[e + f*x])*Sqrt[c + d*Sin[e + f*
x]]),x]
```

output
```
Integrate[Sqrt[g*Sin[e + f*x]]/((a + b*Sin[e + f*x])*Sqrt[c + d*Sin[e + f*
x]]), x]
```

3.47.3 Rubi [A] (verified)

Time = 0.43 (sec) , antiderivative size = 114, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.051, Rules used = {3042, 3416}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sqrt{g \sin(e+fx)}}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}dx$$
$$\Big\downarrow \text{3042}$$
$$\int \frac{\sqrt{g \sin(e+fx)}}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}dx$$
$$\Big\downarrow \text{3416}$$
$$\frac{2\tan(e+fx)\sqrt{-\cot^2(e+fx)}\sqrt{g\sin(e+fx)}\sqrt{\frac{c\csc(e+fx)+d}{c+d}}\,\text{EllipticPi}\left(\frac{2a}{a+b},\arcsin\left(\frac{\sqrt{1-\csc(e+fx)}}{\sqrt{2}}\right),\frac{2c}{c+d}\right)}{f(a+b)\sqrt{c+d\sin(e+fx)}}$$

input
```
Int[Sqrt[g*Sin[e + f*x]]/((a + b*Sin[e + f*x])*Sqrt[c + d*Sin[e + f*x]]),x
]
```

output
```
(2*Sqrt[-Cot[e + f*x]^2]*Sqrt[(d + c*Csc[e + f*x])/(c + d)]*EllipticPi[(2*
a)/(a + b), ArcSin[Sqrt[1 - Csc[e + f*x]]/Sqrt[2]], (2*c)/(c + d)]*Sqrt[g*
Sin[e + f*x]]*Tan[e + f*x])/((a + b)*f*Sqrt[c + d*Sin[e + f*x]])
```

3.47.3.1 Defintions of rubi rules used

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3416
```
Int[Sqrt[(g_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.
)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[2*Sqrt[
-Cot[e + f*x]^2]*(Sqrt[g*Sin[e + f*x]]/(f*(c + d)*Cot[e + f*x]*Sqrt[a + b*S
in[e + f*x]]))*Sqrt[(b + a*Csc[e + f*x])/(a + b)]*EllipticPi[2*(c/(c + d)),
 ArcSin[Sqrt[1 - Csc[e + f*x]]/Sqrt[2]], 2*(a/(a + b))], x] /; FreeQ[{a, b,
 c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 -
d^2, 0]
```

3.47.4 Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 2589 vs. $2(107) = 214$.

Time = 2.92 (sec) , antiderivative size = 2590, normalized size of antiderivative = 22.72

method	result	size
default	Expression too large to display	2590

input
```
int((g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))/(c+d*sin(f*x+e))^(1/2),x,method=
_RETURNVERBOSE)
```

output
```
1/f*(2*EllipticPi(((c*csc(f*x+e)-c*cot(f*x+e)+(-c^2+d^2)^(1/2)+d)/((-c^2+d
^2)^(1/2)+d))^(1/2),((-c^2+d^2)^(1/2)+d)*a/(c*(-a^2+b^2)^(1/2)+a*(-c^2+d^2
)^(1/2)+a*d-b*c),1/2*2^(1/2)*(((-c^2+d^2)^(1/2)+d)/(-c^2+d^2)^(1/2))^(1/2)
)*d*(-c^2+d^2)^(1/2)*(-a^2+b^2)^(1/2)+2*EllipticPi(((c*csc(f*x+e)-c*cot(f*
x+e)+(-c^2+d^2)^(1/2)+d)/((-c^2+d^2)^(1/2)+d))^(1/2),-((-c^2+d^2)^(1/2)+d)
*a/(c*(-a^2+b^2)^(1/2)-a*(-c^2+d^2)^(1/2)-a*d+b*c),1/2*2^(1/2)*(((-c^2+d^2
)^(1/2)+d)/(-c^2+d^2)^(1/2))^(1/2))*d*(-c^2+d^2)^(1/2)*(-a^2+b^2)^(1/2)-El
lipticPi(((c*csc(f*x+e)-c*cot(f*x+e)+(-c^2+d^2)^(1/2)+d)/((-c^2+d^2)^(1/2)
+d))^(1/2),((-c^2+d^2)^(1/2)+d)*a/(c*(-a^2+b^2)^(1/2)+a*(-c^2+d^2)^(1/2)+a
*d-b*c),1/2*2^(1/2)*(((-c^2+d^2)^(1/2)+d)/(-c^2+d^2)^(1/2))^(1/2))*c^2*(-a
^2+b^2)^(1/2)+2*EllipticPi(((c*csc(f*x+e)-c*cot(f*x+e)+(-c^2+d^2)^(1/2)+d)
/((-c^2+d^2)^(1/2)+d))^(1/2),((-c^2+d^2)^(1/2)+d)*a/(c*(-a^2+b^2)^(1/2)+a*
(-c^2+d^2)^(1/2)+a*d-b*c),1/2*2^(1/2)*(((-c^2+d^2)^(1/2)+d)/(-c^2+d^2)^(1/
2))^(1/2))*d^2*(-a^2+b^2)^(1/2)-EllipticPi(((c*csc(f*x+e)-c*cot(f*x+e)+(-c
^2+d^2)^(1/2)+d)/((-c^2+d^2)^(1/2)+d))^(1/2),-((-c^2+d^2)^(1/2)+d)*a/(c*(-
a^2+b^2)^(1/2)-a*(-c^2+d^2)^(1/2)-a*d+b*c),1/2*2^(1/2)*(((-c^2+d^2)^(1/2)+
d)/(-c^2+d^2)^(1/2))^(1/2))*c^2*(-a^2+b^2)^(1/2)+2*EllipticPi(((c*csc(f*x+
e)-c*cot(f*x+e)+(-c^2+d^2)^(1/2)+d)/((-c^2+d^2)^(1/2)+d))^(1/2),-((-c^2+d^
2)^(1/2)+d)*a/(c*(-a^2+b^2)^(1/2)-a*(-c^2+d^2)^(1/2)-a*d+b*c),1/2*2^(1/2)*
(((-c^2+d^2)^(1/2)+d)/(-c^2+d^2)^(1/2))^(1/2))*d^2*(-a^2+b^2)^(1/2)+Ell...
```

3.47.5 Fricas [F(-1)]

Timed out.

$$\int \frac{\sqrt{g\sin(e+fx)}}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}\,dx = \text{Timed out}$$

input
```
integrate((g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))/(c+d*sin(f*x+e))^(1/2),x,
algorithm="fricas")
```

output
```
Timed out
```

3.47.6 Sympy [F]

$$\int \frac{\sqrt{g\sin(e+fx)}}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}\,dx = \int \frac{\sqrt{g\sin{(e+fx)}}}{(a+b\sin{(e+fx)})\sqrt{c+d\sin{(e+fx)}}}\,dx$$

input
```
integrate((g*sin(f*x+e))**(1/2)/(a+b*sin(f*x+e))/(c+d*sin(f*x+e))**(1/2),x
)
```

output
```
Integral(sqrt(g*sin(e + f*x))/((a + b*sin(e + f*x))*sqrt(c + d*sin(e + f*x
))), x)
```

3.47.7 Maxima [F]

$$\int \frac{\sqrt{g\sin(e+fx)}}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}\,dx = \int \frac{\sqrt{g\sin{(fx+e)}}}{(b\sin{(fx+e)}+a)\sqrt{d\sin{(fx+e)}+c}}\,dx$$

input
```
integrate((g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))/(c+d*sin(f*x+e))^(1/2),x,
algorithm="maxima")
```

output
```
integrate(sqrt(g*sin(f*x + e))/((b*sin(f*x + e) + a)*sqrt(d*sin(f*x + e) +
 c)), x)
```

3.47.8 Giac [F]

$$\int \frac{\sqrt{g\sin(e+fx)}}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}\,dx = \int \frac{\sqrt{g\sin{(fx+e)}}}{(b\sin{(fx+e)}+a)\sqrt{d\sin{(fx+e)}+c}}\,dx$$

input
```
integrate((g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))/(c+d*sin(f*x+e))^(1/2),x,
algorithm="giac")
```

output
```
integrate(sqrt(g*sin(f*x + e))/((b*sin(f*x + e) + a)*sqrt(d*sin(f*x + e) +
 c)), x)
```

3.47.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{g\sin(e+fx)}}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}}\,dx$$
$$=\int \frac{\sqrt{g\,\sin\left(e+f\,x\right)}}{\left(a+b\,\sin\left(e+f\,x\right)\right)\,\sqrt{c+d\,\sin\left(e+f\,x\right)}}\,dx$$

input
```
int((g*sin(e + f*x))^(1/2)/((a + b*sin(e + f*x))*(c + d*sin(e + f*x))^(1/2
)),x)
```

output
```
int((g*sin(e + f*x))^(1/2)/((a + b*sin(e + f*x))*(c + d*sin(e + f*x))^(1/2
)), x)
```

3.48 $\int \csc(e+fx)\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}\,dx$

3.48.1 Optimal result

Integrand size = 35, antiderivative size = 391

$$\int \csc(e+fx)\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}\,dx = -\frac{2\sqrt{c+d}\,\text{EllipticPi}\left(\frac{a(c+d)}{(a+b)c},\arcsin\left(\frac{\sqrt{a+b}\sqrt{c+d\sin(e+fx)}}{\sqrt{c+d}\sqrt{a+b\sin(e+fx)}}\right),\frac{(a-b)(c+d)}{(a+b)(c-d)}\right)\sec(e+fx)\sqrt{-\frac{(bc-ad)(1-\sin(e+fx))}{(c+d)(a+b\sin(e+fx))}}\sqrt{(\cdots}}{\sqrt{a+b}f} + \frac{2\sqrt{c+d}\,\text{EllipticPi}\left(\frac{b(c+d)}{(a+b)d},\arcsin\left(\frac{\sqrt{a+b}\sqrt{c+d\sin(e+fx)}}{\sqrt{c+d}\sqrt{a+b\sin(e+fx)}}\right),\frac{(a-b)(c+d)}{(a+b)(c-d)}\right)\sec(e+fx)\sqrt{-\frac{(bc-ad)(1-\sin(e+fx))}{(c+d)(a+b\sin(e+fx))}}\sqrt{(\cdots}}{\sqrt{a+b}f}$$

output
```
-2*EllipticPi((a+b)^(1/2)*(c+d*sin(f*x+e))^(1/2)/(c+d)^(1/2)/(a+b*sin(f*x+
e))^(1/2),a*(c+d)/(a+b)/c,((a-b)*(c+d)/(a+b)/(c-d))^(1/2))*sec(f*x+e)*(a+b
*sin(f*x+e))*(c+d)^(1/2)*(-(-a*d+b*c)*(1-sin(f*x+e))/(c+d)/(a+b*sin(f*x+e)
))^(1/2)*((-a*d+b*c)*(1+sin(f*x+e))/(c-d)/(a+b*sin(f*x+e)))^(1/2)/f/(a+b)^
(1/2)+2*EllipticPi((a+b)^(1/2)*(c+d*sin(f*x+e))^(1/2)/(c+d)^(1/2)/(a+b*sin
(f*x+e))^(1/2),b*(c+d)/(a+b)/d,((a-b)*(c+d)/(a+b)/(c-d))^(1/2))*sec(f*x+e)
*(a+b*sin(f*x+e))*(c+d)^(1/2)*(-(-a*d+b*c)*(1-sin(f*x+e))/(c+d)/(a+b*sin(f
*x+e)))^(1/2)*((-a*d+b*c)*(1+sin(f*x+e))/(c-d)/(a+b*sin(f*x+e)))^(1/2)/f/(
a+b)^(1/2)
```

3.48.2 Mathematica [A] (verified)

Time = 0.19 (sec) , antiderivative size = 274, normalized size of antiderivative = 0.70

$$\int \csc(e+fx)\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}\,dx =$$

$$-\frac{2\sqrt{c+d}\left(\text{EllipticPi}\left(\frac{a(c+d)}{(a+b)c},\arcsin\left(\frac{\sqrt{a+b}\sqrt{c+d\sin(e+fx)}}{\sqrt{c+d}\sqrt{a+b\sin(e+fx)}}\right),\frac{(a-b)(c+d)}{(a+b)(c-d)}\right)-\text{EllipticPi}\left(\frac{b(c+d)}{(a+b)d},\arcsin\left(\frac{\sqrt{a+b}\sqrt{c}}{\sqrt{c+d}\sqrt{c}}\right.\right.\right.}{\sqrt{a+b}}$$

input
```
Integrate[Csc[e + f*x]*Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]],x
]
```

output
```
(-2*Sqrt[c + d]*(EllipticPi[(a*(c + d))/((a + b)*c), ArcSin[(Sqrt[a + b]*S
qrt[c + d*Sin[e + f*x]])/(Sqrt[c + d]*Sqrt[a + b*Sin[e + f*x]])], ((a - b)
*(c + d))/((a + b)*(c - d))] - EllipticPi[(b*(c + d))/((a + b)*d), ArcSin[
(Sqrt[a + b]*Sqrt[c + d*Sin[e + f*x]])/(Sqrt[c + d]*Sqrt[a + b*Sin[e + f*x
]])], ((a - b)*(c + d))/((a + b)*(c - d))])*Sec[e + f*x]*Sqrt[((b*c - a*d)
*(-1 + Sin[e + f*x]))/((c + d)*(a + b*Sin[e + f*x]))]*Sqrt[((b*c - a*d)*(1
 + Sin[e + f*x]))/((c - d)*(a + b*Sin[e + f*x]))]*(a + b*Sin[e + f*x]))/(S
qrt[a + b]*f)
```

3.48.3 Rubi [A] (verified)

Time = 1.11 (sec) , antiderivative size = 391, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.143$, Rules used = {3042, 3428, 3042, 3290, 3424}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \csc(e+fx)\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}{\sin(e+fx)}dx$$

$$\Big\downarrow \text{3428}$$

$$d\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}}dx+c\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}}dx$$

$$\Big\downarrow \; 3042$$

$$d\int \frac{\sqrt{a+b\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}}dx+c\int \frac{\sqrt{a+b\sin(e+fx)}}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}}dx$$

$$\Big\downarrow \; 3290$$

$$c\int \frac{\sqrt{a+b\sin(e+fx)}}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}}dx+$$
$$\frac{2\sqrt{c+d}\sec(e+fx)(a+b\sin(e+fx))\sqrt{-\frac{(bc-ad)(1-\sin(e+fx))}{(c+d)(a+b\sin(e+fx))}}\sqrt{\frac{(bc-ad)(\sin(e+fx)+1)}{(c-d)(a+b\sin(e+fx))}}\text{EllipticPi}\left(\frac{b(c+d)}{(a+b)d},\arcsin\left(\frac{\sqrt{a+}}{\sqrt{c+}}\right.\right.}{f\sqrt{a+b}}$$

$$\Big\downarrow \; 3424$$

$$\frac{2\sqrt{c+d}\sec(e+fx)(a+b\sin(e+fx))\sqrt{-\frac{(bc-ad)(1-\sin(e+fx))}{(c+d)(a+b\sin(e+fx))}}\sqrt{\frac{(bc-ad)(\sin(e+fx)+1)}{(c-d)(a+b\sin(e+fx))}}\text{EllipticPi}\left(\frac{b(c+d)}{(a+b)d},\arcsin\left(\frac{\sqrt{a+}}{\sqrt{c+}}\right.\right.}{f\sqrt{a+b}}$$
$$\frac{2\sqrt{c+d}\sec(e+fx)(a+b\sin(e+fx))\sqrt{-\frac{(bc-ad)(1-\sin(e+fx))}{(c+d)(a+b\sin(e+fx))}}\sqrt{\frac{(bc-ad)(\sin(e+fx)+1)}{(c-d)(a+b\sin(e+fx))}}\text{EllipticPi}\left(\frac{a(c+d)}{(a+b)c},\arcsin\left(\frac{\sqrt{a+}}{\sqrt{c+}}\right.\right.}{f\sqrt{a+b}}$$

input
```
Int[Csc[e + f*x]*Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]],x]
```

output
```
(-2*Sqrt[c + d]*EllipticPi[(a*(c + d))/((a + b)*c), ArcSin[(Sqrt[a + b]*Sq
rt[c + d*Sin[e + f*x]])/(Sqrt[c + d]*Sqrt[a + b*Sin[e + f*x]])], ((a - b)*
(c + d))/((a + b)*(c - d))]*Sec[e + f*x]*Sqrt[-(((b*c - a*d)*(1 - Sin[e +
f*x]))/((c + d)*(a + b*Sin[e + f*x])))]*Sqrt[((b*c - a*d)*(1 + Sin[e + f*x
]))/((c - d)*(a + b*Sin[e + f*x]))]*(a + b*Sin[e + f*x]))/(Sqrt[a + b]*f)
+ (2*Sqrt[c + d]*EllipticPi[(b*(c + d))/((a + b)*d), ArcSin[(Sqrt[a + b]*S
qrt[c + d*Sin[e + f*x]])/(Sqrt[c + d]*Sqrt[a + b*Sin[e + f*x]])], ((a - b)
*(c + d))/((a + b)*(c - d))]*Sec[e + f*x]*Sqrt[-(((b*c - a*d)*(1 - Sin[e +
 f*x]))/((c + d)*(a + b*Sin[e + f*x])))]*Sqrt[((b*c - a*d)*(1 + Sin[e + f*
x]))/((c - d)*(a + b*Sin[e + f*x]))]*(a + b*Sin[e + f*x]))/(Sqrt[a + b]*f)
```

3.48. $\int \csc(e+fx)\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}\,dx$

3.48.3.1 Defintions of rubi rules used

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3290
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/Sqrt[(c_.) + (d_.)*sin[(e_.)
 + (f_.)*(x_)]], x_Symbol] :> Simp[2*((a + b*Sin[e + f*x])/(d*f*Rt[(a + b)/
(c + d), 2]*Cos[e + f*x]))*Sqrt[(b*c - a*d)*((1 + Sin[e + f*x])/((c - d)*(a
 + b*Sin[e + f*x])))]*Sqrt[(-(b*c - a*d))*((1 - Sin[e + f*x])/((c + d)*(a +
 b*Sin[e + f*x])))]*EllipticPi[b*((c + d)/(d*(a + b))), ArcSin[Rt[(a + b)/(
c + d), 2]*(Sqrt[c + d*Sin[e + f*x]]/Sqrt[a + b*Sin[e + f*x]])], (a - b)*((
c + d)/((a + b)*(c - d)))], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c -
 a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0] && PosQ[(a + b)/(c + d)]
```

rule 3424
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(sin[(e_.) + (f_.)*(x_)]*Sqr
t[(c_) + (d_.)*sin[(e_.) + (f_.)*(x_)]]), x_Symbol] :> Simp[-2*((a + b*Sin[
e + f*x])/(c*f*Rt[(a + b)/(c + d), 2]*Cos[e + f*x]))*Sqrt[(-(b*c - a*d))*((
1 - Sin[e + f*x])/((c + d)*(a + b*Sin[e + f*x])))]*Sqrt[(b*c - a*d)*((1 + S
in[e + f*x])/((c - d)*(a + b*Sin[e + f*x])))]*EllipticPi[a*((c + d)/(c*(a +
 b))), ArcSin[Rt[(a + b)/(c + d), 2]*(Sqrt[c + d*Sin[e + f*x]]/Sqrt[a + b*S
in[e + f*x]])], (a - b)*((c + d)/((a + b)*(c - d)))], x] /; FreeQ[{a, b, c,
 d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0
]
```

rule 3428
```
Int[(Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(c_) + (d_.)*sin[(e_.)
 + (f_.)*(x_)]])/sin[(e_.) + (f_.)*(x_)], x_Symbol] :> Simp[d   Int[Sqrt[a
+ b*Sin[e + f*x]]/Sqrt[c + d*Sin[e + f*x]], x], x] + Simp[c   Int[Sqrt[a +
b*Sin[e + f*x]]/(Sin[e + f*x]*Sqrt[c + d*Sin[e + f*x]]), x], x] /; FreeQ[{a
, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && (NeQ[a^2 - b^2, 0] || NeQ[c^2
- d^2, 0])
```

3.48.4 Maple [C] (warning: unable to verify)

Result contains complex when optimal does not.

Time = 7.63 (sec) , antiderivative size = 242134, normalized size of antiderivative = 619.27

method	result	size
default	Expression too large to display	242134

input
```
int((a+b*sin(f*x+e))^(1/2)*(c+d*sin(f*x+e))^(1/2)/sin(f*x+e),x,method=_RET
URNVERBOSE)
```

output
```
result too large to display
```

3.48.5 Fricas [F(-1)]

Timed out.

$$\int \csc(e+fx)\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}\,dx = \text{Timed out}$$

input
```
integrate((a+b*sin(f*x+e))^(1/2)*(c+d*sin(f*x+e))^(1/2)/sin(f*x+e),x, algo
rithm="fricas")
```

output
```
Timed out
```

3.48.6 Sympy [F]

$$\int \csc(e+fx)\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}\,dx = \int \frac{\sqrt{a+b\sin{(e+fx)}}\sqrt{c+d\sin{(e+fx)}}}{\sin{(e+fx)}}\,dx$$

input
```
integrate((a+b*sin(f*x+e))**(1/2)*(c+d*sin(f*x+e))**(1/2)/sin(f*x+e),x)
```

output
```
Integral(sqrt(a + b*sin(e + f*x))*sqrt(c + d*sin(e + f*x))/sin(e + f*x), x
)
```

3.48.7 Maxima [F]

$$\int \csc(e+fx)\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}\,dx = \int \frac{\sqrt{b\sin(fx+e)+a}\sqrt{d\sin(fx+e)+c}}{\sin(fx+e)}\,dx$$

input
```
integrate((a+b*sin(f*x+e))^(1/2)*(c+d*sin(f*x+e))^(1/2)/sin(f*x+e),x, algo
rithm="maxima")
```

output
```
integrate(sqrt(b*sin(f*x + e) + a)*sqrt(d*sin(f*x + e) + c)/sin(f*x + e),
x)
```

3.48.8 Giac [F]

$$\int \csc(e+fx)\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}\,dx = \int \frac{\sqrt{b\sin(fx+e)+a}\sqrt{d\sin(fx+e)+c}}{\sin(fx+e)}\,dx$$

input
```
integrate((a+b*sin(f*x+e))^(1/2)*(c+d*sin(f*x+e))^(1/2)/sin(f*x+e),x, algo
rithm="giac")
```

output
```
integrate(sqrt(b*sin(f*x + e) + a)*sqrt(d*sin(f*x + e) + c)/sin(f*x + e),
x)
```

3.48.9 Mupad [F(-1)]

Timed out.

$$\int \csc(e+fx)\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}\,dx = \int \frac{\sqrt{a+b\,\sin(e+f\,x)}\,\sqrt{c+d\,\sin(e+f\,x)}}{\sin(e+f\,x)}\,dx$$

input
```
int(((a + b*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))^(1/2))/sin(e + f*x),x
)
```

output
```
int(((a + b*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))^(1/2))/sin(e + f*x),
x)
```

3.48. $\int \csc(e+fx)\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}\,dx$

3.49 $\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}}\,dx$

3.49.1 Optimal result

Integrand size = 35, antiderivative size = 198

$$\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}}\,dx = -\frac{2\sqrt{c+d}\,\text{EllipticPi}\left(\frac{a(c+d)}{(a+b)c},\arcsin\left(\frac{\sqrt{a+b}\sqrt{c+d\sin(e+fx)}}{\sqrt{c+d}\sqrt{a+b\sin(e+fx)}}\right),\frac{(a-b)(c+d)}{(a+b)(c-d)}\right)\sec(e+fx)\sqrt{-\frac{(bc-ad)(1-\sin(e+fx))}{(c+d)(a+b\sin(e+fx))}}\sqrt{(\cdots}}{\sqrt{a+b}cf}$$

output
```
-2*EllipticPi((a+b)^(1/2)*(c+d*sin(f*x+e))^(1/2)/(c+d)^(1/2)/(a+b*sin(f*x+
e))^(1/2),a*(c+d)/(a+b)/c,((a-b)*(c+d)/(a+b)/(c-d))^(1/2))*sec(f*x+e)*(a+b
*sin(f*x+e))*(c+d)^(1/2)*(-(-a*d+b*c)*(1-sin(f*x+e))/(c+d)/(a+b*sin(f*x+e)
))^(1/2)*((-a*d+b*c)*(1+sin(f*x+e))/(c-d)/(a+b*sin(f*x+e)))^(1/2)/c/f/(a+b
)^(1/2)
```

3.49.2 Mathematica [A] (verified)

Time = 0.11 (sec) , antiderivative size = 197, normalized size of antiderivative = 0.99

$$\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}}\,dx = -\frac{2\sqrt{c+d}\,\text{EllipticPi}\left(\frac{a(c+d)}{(a+b)c},\arcsin\left(\frac{\sqrt{a+b}\sqrt{c+d\sin(e+fx)}}{\sqrt{c+d}\sqrt{a+b\sin(e+fx)}}\right),\frac{(a-b)(c+d)}{(a+b)(c-d)}\right)\sec(e+fx)\sqrt{\frac{(-bc+ad)(1-\sin(e+fx))}{(c+d)(a+b\sin(e+fx))}}\sqrt{\frac{(\cdots}{(\cdots}}}{\sqrt{a+b}cf}$$

input
```
Integrate[(Csc[e + f*x]*Sqrt[a + b*Sin[e + f*x]])/Sqrt[c + d*Sin[e + f*x]]
,x]
```

output
```
(-2*Sqrt[c + d]*EllipticPi[(a*(c + d))/((a + b)*c), ArcSin[(Sqrt[a + b]*Sq
rt[c + d*Sin[e + f*x]])/(Sqrt[c + d]*Sqrt[a + b*Sin[e + f*x]])], ((a - b)*
(c + d))/((a + b)*(c - d))]*Sec[e + f*x]*Sqrt[((-(b*c) + a*d)*(1 - Sin[e +
 f*x]))/((c + d)*(a + b*Sin[e + f*x]))]*Sqrt[((b*c - a*d)*(1 + Sin[e + f*x
]))/((c - d)*(a + b*Sin[e + f*x]))]*(a + b*Sin[e + f*x]))/(Sqrt[a + b]*c*f
)
```

3.49.3 Rubi [A] (verified)

Time = 0.45 (sec) , antiderivative size = 198, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.057, Rules used = {3042, 3424}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\sqrt{a+b\sin(e+fx)}}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}}dx$$

$$\Big\downarrow \text{3424}$$

$$-\frac{2\sqrt{c+d}\sec(e+fx)(a+b\sin(e+fx))\sqrt{-\frac{(bc-ad)(1-\sin(e+fx))}{(c+d)(a+b\sin(e+fx))}}\sqrt{\frac{(bc-ad)(\sin(e+fx)+1)}{(c-d)(a+b\sin(e+fx))}}\text{EllipticPi}\left(\frac{a(c+d)}{(a+b)c},\arcsin\left(\frac{\sqrt{}}{\sqrt{}}\right.\right.}{cf\sqrt{a+b}}$$

input
```
Int[(Csc[e + f*x]*Sqrt[a + b*Sin[e + f*x]])/Sqrt[c + d*Sin[e + f*x]],x]
```

output
```
(-2*Sqrt[c + d]*EllipticPi[(a*(c + d))/((a + b)*c), ArcSin[(Sqrt[a + b]*Sq
rt[c + d*Sin[e + f*x]])/(Sqrt[c + d]*Sqrt[a + b*Sin[e + f*x]])], ((a - b)*
(c + d))/((a + b)*(c - d))]*Sec[e + f*x]*Sqrt[-(((b*c - a*d)*(1 - Sin[e +
f*x]))/((c + d)*(a + b*Sin[e + f*x])))]*Sqrt[((b*c - a*d)*(1 + Sin[e + f*x
]))/((c - d)*(a + b*Sin[e + f*x]))]*(a + b*Sin[e + f*x]))/(Sqrt[a + b]*c*f
)
```

3.49.3.1 Defintions of rubi rules used

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3424
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(sin[(e_.) + (f_.)*(x_)]*Sqr
t[(c_) + (d_.)*sin[(e_.) + (f_.)*(x_)]]), x_Symbol] :> Simp[-2*((a + b*Sin[
e + f*x])/(c*f*Rt[(a + b)/(c + d), 2]*Cos[e + f*x]))*Sqrt[(-(b*c - a*d))*((
1 - Sin[e + f*x])/((c + d)*(a + b*Sin[e + f*x])))]*Sqrt[(b*c - a*d)*((1 + S
in[e + f*x])/((c - d)*(a + b*Sin[e + f*x])))]*EllipticPi[a*((c + d)/(c*(a +
 b))), ArcSin[Rt[(a + b)/(c + d), 2]*(Sqrt[c + d*Sin[e + f*x]]/Sqrt[a + b*S
in[e + f*x]])], (a - b)*((c + d)/((a + b)*(c - d)))], x] /; FreeQ[{a, b, c,
 d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0
]
```

3.49.4 Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 28725 vs. 2(183) = 366.

Time = 7.15 (sec) , antiderivative size = 28726, normalized size of antiderivative = 145.08

method	result	size
default	Expression too large to display	28726

input
```
int((a+b*sin(f*x+e))^(1/2)/sin(f*x+e)/(c+d*sin(f*x+e))^(1/2),x,method=_RET
URNVERBOSE)
```

output
```
result too large to display
```

3.49.5 Fricas [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}}\,dx = \text{Timed out}$$

input
```
integrate((a+b*sin(f*x+e))^(1/2)/sin(f*x+e)/(c+d*sin(f*x+e))^(1/2),x, algo
rithm="fricas")
```

output
```
Timed out
```

3.49.6 Sympy [F]

$$\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}}\,dx = \int \frac{\sqrt{a+b\sin{(e+fx)}}}{\sqrt{c+d\sin{(e+fx)}}\sin{(e+fx)}}\,dx$$

input
```
integrate((a+b*sin(f*x+e))**(1/2)/sin(f*x+e)/(c+d*sin(f*x+e))**(1/2),x)
```

output
```
Integral(sqrt(a + b*sin(e + f*x))/(sqrt(c + d*sin(e + f*x))*sin(e + f*x)),
 x)
```

3.49.7 Maxima [F]

$$\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}}\,dx = \int \frac{\sqrt{b\sin{(fx+e)}+a}}{\sqrt{d\sin{(fx+e)}+c}\sin{(fx+e)}}\,dx$$

input
```
integrate((a+b*sin(f*x+e))^(1/2)/sin(f*x+e)/(c+d*sin(f*x+e))^(1/2),x, algo
rithm="maxima")
```

output
```
integrate(sqrt(b*sin(f*x + e) + a)/(sqrt(d*sin(f*x + e) + c)*sin(f*x + e))
, x)
```

3.49.8 Giac [F]

$$\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}}\,dx = \int \frac{\sqrt{b\sin{(fx+e)}+a}}{\sqrt{d\sin{(fx+e)}+c}\sin{(fx+e)}}\,dx$$

input
```
integrate((a+b*sin(f*x+e))^(1/2)/sin(f*x+e)/(c+d*sin(f*x+e))^(1/2),x, algo
rithm="giac")
```

output
```
integrate(sqrt(b*sin(f*x + e) + a)/(sqrt(d*sin(f*x + e) + c)*sin(f*x + e))
, x)
```

3.49.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}}\,dx = \int \frac{\sqrt{a+b\,\sin\left(e+f\,x\right)}}{\sin\left(e+f\,x\right)\,\sqrt{c+d\,\sin\left(e+f\,x\right)}}\,dx$$

input
```
int((a + b*sin(e + f*x))^(1/2)/(sin(e + f*x)*(c + d*sin(e + f*x))^(1/2)),x
)
```

output
```
int((a + b*sin(e + f*x))^(1/2)/(sin(e + f*x)*(c + d*sin(e + f*x))^(1/2)),
x)
```

3.50 $\int \frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\,dx$

3.50.1 Optimal result

Integrand size = 35, antiderivative size = 398

$$\int \frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\,dx = -\frac{2\sqrt{c+d}\,\text{EllipticPi}\left(\frac{a(c+d)}{(a+b)c},\arcsin\left(\frac{\sqrt{a+b}\sqrt{c+d\sin(e+fx)}}{\sqrt{c+d}\sqrt{a+b\sin(e+fx)}}\right),\frac{(a-b)(c+d)}{(a+b)(c-d)}\right)\sec(e+fx)\sqrt{-\frac{(bc-ad)(1-\sin(e+fx))}{(c+d)(a+b\sin(e+fx))}}\sqrt{\cdots}}{a\sqrt{a+b}cf} - \frac{2b\sqrt{a+b}\,\text{EllipticF}\left(\arcsin\left(\frac{\sqrt{c+d}\sqrt{a+b\sin(e+fx)}}{\sqrt{a+b}\sqrt{c+d\sin(e+fx)}}\right),\frac{(a+b)(c-d)}{(a-b)(c+d)}\right)\sec(e+fx)\sqrt{\frac{(bc-ad)(1-\sin(e+fx))}{(a+b)(c+d\sin(e+fx))}}\sqrt{-\frac{(bc-ad)(\cdots}{(a-b)(c+\cdots}}}{a\sqrt{c+d}(bc-ad)f}$$

output
```
-2*EllipticPi((a+b)^(1/2)*(c+d*sin(f*x+e))^(1/2)/(c+d)^(1/2)/(a+b*sin(f*x+
e))^(1/2),a*(c+d)/(a+b)/c,((a-b)*(c+d)/(a+b)/(c-d))^(1/2))*sec(f*x+e)*(a+b
*sin(f*x+e))*(c+d)^(1/2)*(-(-a*d+b*c)*(1-sin(f*x+e))/(c+d)/(a+b*sin(f*x+e)
))^(1/2)*((-a*d+b*c)*(1+sin(f*x+e))/(c-d)/(a+b*sin(f*x+e)))^(1/2)/a/c/f/(a
+b)^(1/2)-2*b*EllipticF((c+d)^(1/2)*(a+b*sin(f*x+e))^(1/2)/(a+b)^(1/2)/(c+
d*sin(f*x+e))^(1/2),((a+b)*(c-d)/(a-b)/(c+d))^(1/2))*sec(f*x+e)*(c+d*sin(f
*x+e))*(a+b)^(1/2)*((-a*d+b*c)*(1-sin(f*x+e))/(a+b)/(c+d*sin(f*x+e)))^(1/2
)*(-(-a*d+b*c)*(1+sin(f*x+e))/(a-b)/(c+d*sin(f*x+e)))^(1/2)/a/(-a*d+b*c)/f
/(c+d)^(1/2)
```

3.50.2 Mathematica [A] (verified)

Time = 1.62 (sec) , antiderivative size = 374, normalized size of antiderivative = 0.94

$$\int \frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\,dx$$
$$= \frac{2\sec(e+fx)\left(-\frac{(c+d)\,\text{EllipticPi}\left(\frac{a(c+d)}{(a+b)c},\arcsin\left(\frac{\sqrt{a+b}\sqrt{c+d\sin(e+fx)}}{\sqrt{c+d}\sqrt{a+b\sin(e+fx)}}\right),\frac{(a-b)(c+d)}{(a+b)(c-d)}\right)\sqrt{\frac{(bc-ad)(-1+\sin(e+fx))}{(c+d)(a+b\sin(e+fx))}}\sqrt{\frac{(bc-ad)(1+\sin(e+fx))}{(c-d)(a+b\sin(e+fx))}}(a+}{c}}{a\sqrt{a+}$$

input
```
Integrate[Csc[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]])
,x]
```

output
```
(2*Sec[e + f*x]*(-(((c + d)*EllipticPi[(a*(c + d))/((a + b)*c), ArcSin[(Sq
rt[a + b]*Sqrt[c + d*Sin[e + f*x]])/(Sqrt[c + d]*Sqrt[a + b*Sin[e + f*x]])
], ((a - b)*(c + d))/((a + b)*(c - d))]*Sqrt[((b*c - a*d)*(-1 + Sin[e + f*
x]))/((c + d)*(a + b*Sin[e + f*x]))]*Sqrt[((b*c - a*d)*(1 + Sin[e + f*x]))
/((c - d)*(a + b*Sin[e + f*x]))]*(a + b*Sin[e + f*x]))/c) - (b*(a + b)*Ell
ipticF[ArcSin[(Sqrt[c + d]*Sqrt[a + b*Sin[e + f*x]])/(Sqrt[a + b]*Sqrt[c +
 d*Sin[e + f*x]])], ((a + b)*(c - d))/((a - b)*(c + d))]*Sqrt[((-(b*c) + a
*d)*(-1 + Sin[e + f*x]))/((a + b)*(c + d*Sin[e + f*x]))]*Sqrt[((-(b*c) + a
*d)*(1 + Sin[e + f*x]))/((a - b)*(c + d*Sin[e + f*x]))]*(c + d*Sin[e + f*x
]))/(b*c - a*d)))/(a*Sqrt[a + b]*Sqrt[c + d]*f)
```

3.50.3 Rubi [A] (verified)

Time = 1.10 (sec) , antiderivative size = 398, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.143$, Rules used = {3042, 3426, 3042, 3297, 3424}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\,dx$$
$$\Big\downarrow \; 3042$$
$$\int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}dx$$

$$\Big\downarrow \; 3426$$

$$\frac{\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}}dx}{a}-\frac{b\int \frac{1}{\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}dx}{a}$$

$$\Big\downarrow \; 3042$$

$$\frac{\int \frac{\sqrt{a+b\sin(e+fx)}}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}}dx}{a}-\frac{b\int \frac{1}{\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}dx}{a}$$

$$\Big\downarrow \; 3297$$

$$\frac{\int \frac{\sqrt{a+b\sin(e+fx)}}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}}dx}{a}-\frac{2b\sqrt{a+b}\sec(e+fx)(c+d\sin(e+fx))\sqrt{\frac{(bc-ad)(1-\sin(e+fx))}{(a+b)(c+d\sin(e+fx))}}\sqrt{-\frac{(bc-ad)(\sin(e+fx)+1)}{(a-b)(c+d\sin(e+fx))}}\text{EllipticF}\left(\arcsin\left(\frac{\sqrt{c+d}\sqrt{a+b s}}{\sqrt{a+b}\sqrt{c+d s}}\right)\right)}{af\sqrt{c+d}(bc-ad)}$$

$$\Big\downarrow \; 3424$$

$$-\frac{2b\sqrt{a+b}\sec(e+fx)(c+d\sin(e+fx))\sqrt{\frac{(bc-ad)(1-\sin(e+fx))}{(a+b)(c+d\sin(e+fx))}}\sqrt{-\frac{(bc-ad)(\sin(e+fx)+1)}{(a-b)(c+d\sin(e+fx))}}\text{EllipticF}\left(\arcsin\left(\frac{\sqrt{c+d}\sqrt{a+}}{\sqrt{a+b}\sqrt{c+}}\right)\right)}{af\sqrt{c+d}(bc-ad)}$$
$$\frac{2\sqrt{c+d}\sec(e+fx)(a+b\sin(e+fx))\sqrt{-\frac{(bc-ad)(1-\sin(e+fx))}{(c+d)(a+b\sin(e+fx))}}\sqrt{\frac{(bc-ad)(\sin(e+fx)+1)}{(c-d)(a+b\sin(e+fx))}}\text{EllipticPi}\left(\frac{a(c+d)}{(a+b)c},\arcsin\left(\frac{\sqrt{a+}}{\sqrt{c+}}\right)\right)}{acf\sqrt{a+b}}$$

input
```
Int[Csc[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]]),x]
```

output
```
(-2*Sqrt[c + d]*EllipticPi[(a*(c + d))/((a + b)*c), ArcSin[(Sqrt[a + b]*Sq
rt[c + d*Sin[e + f*x]])/(Sqrt[c + d]*Sqrt[a + b*Sin[e + f*x]])], ((a - b)*
(c + d))/((a + b)*(c - d))]*Sec[e + f*x]*Sqrt[-(((b*c - a*d)*(1 - Sin[e +
f*x]))/((c + d)*(a + b*Sin[e + f*x])))]*Sqrt[((b*c - a*d)*(1 + Sin[e + f*x
]))/((c - d)*(a + b*Sin[e + f*x]))]*(a + b*Sin[e + f*x]))/(a*Sqrt[a + b]*c
*f) - (2*b*Sqrt[a + b]*EllipticF[ArcSin[(Sqrt[c + d]*Sqrt[a + b*Sin[e + f*
x]])/(Sqrt[a + b]*Sqrt[c + d*Sin[e + f*x]])], ((a + b)*(c - d))/((a - b)*(
c + d))]*Sec[e + f*x]*Sqrt[((b*c - a*d)*(1 - Sin[e + f*x]))/((a + b)*(c +
d*Sin[e + f*x]))]*Sqrt[-(((b*c - a*d)*(1 + Sin[e + f*x]))/((a - b)*(c + d*
Sin[e + f*x])))]*(c + d*Sin[e + f*x]))/(a*Sqrt[c + d]*(b*c - a*d)*f)
```

3.50. $\int \frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}dx$

3.50.3.1 Defintions of rubi rules used

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3297
```
Int[1/(Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(c_) + (d_.)*sin[(e_
.) + (f_.)*(x_)]]), x_Symbol] :> Simp[2*((c + d*Sin[e + f*x])/(f*(b*c - a*d
)*Rt[(c + d)/(a + b), 2]*Cos[e + f*x]))*Sqrt[(b*c - a*d)*((1 - Sin[e + f*x]
)/((a + b)*(c + d*Sin[e + f*x])))]*Sqrt[(-(b*c - a*d))*((1 + Sin[e + f*x])/
((a - b)*(c + d*Sin[e + f*x])))]*EllipticF[ArcSin[Rt[(c + d)/(a + b), 2]*(S
qrt[a + b*Sin[e + f*x]]/Sqrt[c + d*Sin[e + f*x]])], (a + b)*((c - d)/((a -
b)*(c + d)))], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && N
eQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0] && PosQ[(c + d)/(a + b)]
```

rule 3424
```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(sin[(e_.) + (f_.)*(x_)]*Sqr
t[(c_) + (d_.)*sin[(e_.) + (f_.)*(x_)]]), x_Symbol] :> Simp[-2*((a + b*Sin[
e + f*x])/(c*f*Rt[(a + b)/(c + d), 2]*Cos[e + f*x]))*Sqrt[(-(b*c - a*d))*((
1 - Sin[e + f*x])/((c + d)*(a + b*Sin[e + f*x])))]*Sqrt[(b*c - a*d)*((1 + S
in[e + f*x])/((c - d)*(a + b*Sin[e + f*x])))]*EllipticPi[a*((c + d)/(c*(a +
 b))), ArcSin[Rt[(a + b)/(c + d), 2]*(Sqrt[c + d*Sin[e + f*x]]/Sqrt[a + b*S
in[e + f*x]])], (a - b)*((c + d)/((a + b)*(c - d)))], x] /; FreeQ[{a, b, c,
 d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0
]
```

rule 3426
```
Int[1/(sin[(e_.) + (f_.)*(x_)]*Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]*S
qrt[(c_) + (d_.)*sin[(e_.) + (f_.)*(x_)]]), x_Symbol] :> Simp[-b/a   Int[1/
(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]]), x], x] + Simp[1/a   In
t[Sqrt[a + b*Sin[e + f*x]]/(Sin[e + f*x]*Sqrt[c + d*Sin[e + f*x]]), x], x]
/; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && (NeQ[a^2 - b^2, 0]
|| NeQ[c^2 - d^2, 0])
```

3.50.4 Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 24289 vs. $2(368) = 736$.

Time = 7.12 (sec) , antiderivative size = 24290, normalized size of antiderivative = 61.03

method	result	size
default	Expression too large to display	24290

input
```
int(1/sin(f*x+e)/(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x,method=_R
ETURNVERBOSE)
```

output
```
result too large to display
```

3.50.5 Fricas [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\,dx = \text{Timed out}$$

input
```
integrate(1/sin(f*x+e)/(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x, al
gorithm="fricas")
```

output
```
Timed out
```

3.50.6 Sympy [F]

$$\int \frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\,dx = \int \frac{1}{\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}\sin(e+fx)}\,dx$$

input
```
integrate(1/sin(f*x+e)/(a+b*sin(f*x+e))**(1/2)/(c+d*sin(f*x+e))**(1/2),x)
```

output
```
Integral(1/(sqrt(a + b*sin(e + f*x))*sqrt(c + d*sin(e + f*x))*sin(e + f*x)
), x)
```

3.50.7 Maxima [F]

$$\int \frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\,dx = \int \frac{1}{\sqrt{b\sin\left(fx+e\right)+a}\sqrt{d\sin\left(fx+e\right)+c}\sin\left(fx+e\right)}\,dx$$

input
```
integrate(1/sin(f*x+e)/(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x, al
gorithm="maxima")
```

output
```
integrate(1/(sqrt(b*sin(f*x + e) + a)*sqrt(d*sin(f*x + e) + c)*sin(f*x + e
)), x)
```

3.50.8 Giac [F]

$$\int \frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\,dx = \int \frac{1}{\sqrt{b\sin\left(fx+e\right)+a}\sqrt{d\sin\left(fx+e\right)+c}\sin\left(fx+e\right)}\,dx$$

input
```
integrate(1/sin(f*x+e)/(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x, al
gorithm="giac")
```

output
```
integrate(1/(sqrt(b*sin(f*x + e) + a)*sqrt(d*sin(f*x + e) + c)*sin(f*x + e
)), x)
```

3.50.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\,dx = \int \frac{1}{\sin\left(e+f\,x\right)\,\sqrt{a+b\,\sin\left(e+f\,x\right)}\,\sqrt{c+d\,\sin\left(e+f\,x\right)}}\,dx$$

input
```
int(1/(sin(e + f*x)*(a + b*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))^(1/2))
,x)
```

output
```
int(1/(sin(e + f*x)*(a + b*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))^(1/2))
, x)
```

3.50. $\int \frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\,dx$

3.51 $\int (a+a\sin(e+fx))^m(A+B\sin(e+fx))^p(c-c\sin(e+fx))^n\,dx$

3.51.1 Optimal result

Integrand size = 38, antiderivative size = 157

$$\int (a+a\sin(e+fx))^m(A+B\sin(e+fx))^p(c-c\sin(e+fx))^n\,dx$$
$$=\frac{2^{\frac{1}{2}+n}\operatorname{AppellF1}\left(\frac{1}{2}+m,\frac{1}{2}-n,-p,\frac{3}{2}+m,\frac{1}{2}(1+\sin(e+fx)),-\frac{B(1+\sin(e+fx))}{A-B}\right)\sec(e+fx)(1-\sin(e+f}{af(1+2m)}$$

output
```
2^(1/2+n)*AppellF1(1/2+m,-p,1/2-n,3/2+m,-B*(1+sin(f*x+e))/(A-B),1/2+1/2*si
n(f*x+e))*sec(f*x+e)*(1-sin(f*x+e))^(1/2-n)*(a+a*sin(f*x+e))^(1+m)*(A+B*si
n(f*x+e))^p*(c-c*sin(f*x+e))^n/a/f/(1+2*m)/(((A+B*sin(f*x+e))/(A-B))^p)
```

3.51.2 Mathematica [F]

$$\int (a+a\sin(e+fx))^m(A+B\sin(e+fx))^p(c-c\sin(e+fx))^n\,dx$$
$$=\int (a+a\sin(e+fx))^m(A+B\sin(e+fx))^p(c-c\sin(e+fx))^n\,dx$$

input
```
Integrate[(a + a*Sin[e + f*x])^m*(A + B*Sin[e + f*x])^p*(c - c*Sin[e + f*x
])^n,x]
```

output
```
Integrate[(a + a*Sin[e + f*x])^m*(A + B*Sin[e + f*x])^p*(c - c*Sin[e + f*x
])^n, x]
```

3.51.3 Rubi [A] (verified)

Time = 0.46 (sec) , antiderivative size = 157, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.132$, Rules used = {3042, 3487, 157, 156, 155}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int (a\sin(e+fx)+a)^m(c-c\sin(e+fx))^n(A+B\sin(e+fx))^p\,dx$$

$$\Big\downarrow \; 3042$$

$$\int (a\sin(e+fx)+a)^m(c-c\sin(e+fx))^n(A+B\sin(e+fx))^pdx$$

$$\Big\downarrow \; 3487$$

$$\frac{\sec(e+fx)\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}\int(\sin(e+fx)a+a)^{m-\frac{1}{2}}(A+B\sin(e+fx))^p(c-c\sin(e+fx)}{f}$$

$$\Big\downarrow \; 157$$

$$\frac{\sec(e+fx)\sqrt{a\sin(e+fx)+a}\sqrt{c-c\sin(e+fx)}(A+B\sin(e+fx))^p\left(\frac{A+B\sin(e+fx)}{A-B}\right)^{-p}\int(\sin(e+fx)a+a)^{m-}}{f}$$

$$\Big\downarrow \; 156$$

$$\frac{2^{n-\frac{1}{2}}\sec(e+fx)\sqrt{a\sin(e+fx)+a}(1-\sin(e+fx))^{\frac{1}{2}-n}(c-c\sin(e+fx))^n(A+B\sin(e+fx))^p\left(\frac{A+B\sin(e+fx)}{A-B}\right.}{f}$$

$$\Big\downarrow \; 155$$

$$\frac{2^{n+\frac{1}{2}}\sec(e+fx)(1-\sin(e+fx))^{\frac{1}{2}-n}(a\sin(e+fx)+a)^{m+1}(c-c\sin(e+fx))^n(A+B\sin(e+fx))^p\left(\frac{A+B\sin(e-}{A-B}\right.}{af(2m+1)}$$

input
```
Int[(a + a*Sin[e + f*x])^m*(A + B*Sin[e + f*x])^p*(c - c*Sin[e + f*x])^n,x
]
```

output
```
(2^(1/2 + n)*AppellF1[1/2 + m, 1/2 - n, -p, 3/2 + m, (1 + Sin[e + f*x])/2,
 -((B*(1 + Sin[e + f*x]))/(A - B))]*Sec[e + f*x]*(1 - Sin[e + f*x])^(1/2 -
 n)*(a + a*Sin[e + f*x])^(1 + m)*(A + B*Sin[e + f*x])^p*(c - c*Sin[e + f*x
])^n)/(a*f*(1 + 2*m)*((A + B*Sin[e + f*x])/(A - B))^p)
```

3.51.3.1 Defintions of rubi rules used

rule 155
```
Int[((a_) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_)*((e_.) + (f_.)*(x_))
^(p_), x_] :> Simp[((a + b*x)^(m + 1)/(b*(m + 1)*Simplify[b/(b*c - a*d)]^n*
Simplify[b/(b*e - a*f)]^p))*AppellF1[m + 1, -n, -p, m + 2, (-d)*((a + b*x)/
(b*c - a*d)), (-f)*((a + b*x)/(b*e - a*f))], x] /; FreeQ[{a, b, c, d, e, f,
 m, n, p}, x] && !IntegerQ[m] && !IntegerQ[n] && !IntegerQ[p] && GtQ[Sim
plify[b/(b*c - a*d)], 0] && GtQ[Simplify[b/(b*e - a*f)], 0] && !(GtQ[Simpl
ify[d/(d*a - c*b)], 0] && GtQ[Simplify[d/(d*e - c*f)], 0] && SimplerQ[c + d
*x, a + b*x]) && !(GtQ[Simplify[f/(f*a - e*b)], 0] && GtQ[Simplify[f/(f*c
- e*d)], 0] && SimplerQ[e + f*x, a + b*x])
```

rule 156
```
Int[((a_) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_)*((e_.) + (f_.)*(x_))
^(p_), x_] :> Simp[(e + f*x)^FracPart[p]/(Simplify[b/(b*e - a*f)]^IntPart[p
]*(b*((e + f*x)/(b*e - a*f)))^FracPart[p])   Int[(a + b*x)^m*(c + d*x)^n*Si
mp[b*(e/(b*e - a*f)) + b*f*(x/(b*e - a*f)), x]^p, x], x] /; FreeQ[{a, b, c,
 d, e, f, m, n, p}, x] && !IntegerQ[m] && !IntegerQ[n] && !IntegerQ[p] &
& GtQ[Simplify[b/(b*c - a*d)], 0] && !GtQ[Simplify[b/(b*e - a*f)], 0]
```

rule 157
```
Int[((a_) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_)*((e_.) + (f_.)*(x_))
^(p_), x_] :> Simp[(c + d*x)^FracPart[n]/(Simplify[b/(b*c - a*d)]^IntPart[n
]*(b*((c + d*x)/(b*c - a*d)))^FracPart[n])   Int[(a + b*x)^m*Simp[b*(c/(b*c
 - a*d)) + b*d*(x/(b*c - a*d)), x]^n*(e + f*x)^p, x], x] /; FreeQ[{a, b, c,
 d, e, f, m, n, p}, x] && !IntegerQ[m] && !IntegerQ[n] && !IntegerQ[p] &
&  !GtQ[Simplify[b/(b*c - a*d)], 0] && !SimplerQ[c + d*x, a + b*x] && !Si
mplerQ[e + f*x, a + b*x]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3487
```
Int[((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_.)*((A_.) + (B_.)*sin[(e_.) +
 (f_.)*(x_)])^(p_)*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])^(n_.), x_Symbol]
:> Simp[Sqrt[a + b*Sin[e + f*x]]*(Sqrt[c + d*Sin[e + f*x]]/(f*Cos[e + f*x])
)   Subst[Int[(a + b*x)^(m - 1/2)*(c + d*x)^(n - 1/2)*(A + B*x)^p, x], x, S
in[e + f*x]], x] /; FreeQ[{a, b, c, d, e, f, A, B, m, n, p}, x] && EqQ[b*c
+ a*d, 0] && EqQ[a^2 - b^2, 0]
```

3.51.4 Maple [F]

$$\int (a+a\sin (fx+e))^m (A+B\sin (fx+e))^p (c-c\sin (fx+e))^n\,dx$$

input
```
int((a+a*sin(f*x+e))^m*(A+B*sin(f*x+e))^p*(c-c*sin(f*x+e))^n,x)
```

output
```
int((a+a*sin(f*x+e))^m*(A+B*sin(f*x+e))^p*(c-c*sin(f*x+e))^n,x)
```

3.51.5 Fricas [F]

$$\int (a+a\sin(e+fx))^m(A+B\sin(e+fx))^p(c-c\sin(e+fx))^n\,dx$$
$$=\int (B\sin (fx+e)+A)^p(a\sin (fx+e)+a)^m(-c\sin (fx+e)+c)^n\,dx$$

input
```
integrate((a+a*sin(f*x+e))^m*(A+B*sin(f*x+e))^p*(c-c*sin(f*x+e))^n,x, algo
rithm="fricas")
```

output
```
integral((B*sin(f*x + e) + A)^p*(a*sin(f*x + e) + a)^m*(-c*sin(f*x + e) +
c)^n, x)
```

3.51.6 Sympy [F(-1)]

Timed out.

$$\int (a + a\sin(e + fx))^m (A + B\sin(e + fx))^p (c - c\sin(e + fx))^n \, dx = \text{Timed out}$$

input
```
integrate((a+a*sin(f*x+e))**m*(A+B*sin(f*x+e))**p*(c-c*sin(f*x+e))**n,x)
```

output
```
Timed out
```

3.51.7 Maxima [F]

$$\int (a + a\sin(e + fx))^m (A + B\sin(e + fx))^p (c - c\sin(e + fx))^n \, dx$$
$$= \int (B\sin(fx + e) + A)^p (a\sin(fx + e) + a)^m (-c\sin(fx + e) + c)^n \, dx$$

input
```
integrate((a+a*sin(f*x+e))^m*(A+B*sin(f*x+e))^p*(c-c*sin(f*x+e))^n,x, algo
rithm="maxima")
```

output
```
integrate((B*sin(f*x + e) + A)^p*(a*sin(f*x + e) + a)^m*(-c*sin(f*x + e) +
 c)^n, x)
```

3.51.8 Giac [F]

$$\int (a + a\sin(e + fx))^m (A + B\sin(e + fx))^p (c - c\sin(e + fx))^n \, dx$$
$$= \int (B\sin(fx + e) + A)^p (a\sin(fx + e) + a)^m (-c\sin(fx + e) + c)^n \, dx$$

input
```
integrate((a+a*sin(f*x+e))^m*(A+B*sin(f*x+e))^p*(c-c*sin(f*x+e))^n,x, algo
rithm="giac")
```

output
```
integrate((B*sin(f*x + e) + A)^p*(a*sin(f*x + e) + a)^m*(-c*sin(f*x + e) +
 c)^n, x)
```

3.51.9 Mupad [F(-1)]

Timed out.

$$\int (a+a\sin(e+fx))^m(A+B\sin(e+fx))^p(c-c\sin(e+fx))^n\,dx$$
$$=\int (A+B\,\sin(e+f\,x))^p\,(a+a\,\sin(e+f\,x))^m\,(c-c\,\sin(e+f\,x))^n\,dx$$

input
```
int((A + B*sin(e + f*x))^p*(a + a*sin(e + f*x))^m*(c - c*sin(e + f*x))^n,x
)
```

output
```
int((A + B*sin(e + f*x))^p*(a + a*sin(e + f*x))^m*(c - c*sin(e + f*x))^n,
x)
```

CHAPTER 4

Appendix

4.1 Listing of Grading functions

The following are the current version of the grading functions used for grading the quality of the antiderivative with reference to the optimal antiderivative included in the test suite.

There is a version for Maple and for Mathematica/Rubi. There is a version for grading Sympy and version for use with Sagemath.

The following are links to the current source code.

The following are the listings of source code of the grading functions.

4.1.1 Mathematica and Rubi grading function

```
(* Original version thanks to Albert Rich emailed on 03/21/2017 *)
(* ::Package:: *)

(* Nasser: April 7,2022. add second output which gives reason for the grade *)
(*                       Small rewrite of logic in main function to make it*)
(*                       match Maple's logic. No change in functionality otherwise*)

(* ::Subsection:: *)
(*GradeAntiderivative[result,optimal]*)


(* ::Text:: *)
(*If result and optimal are mathematical expressions, *)
(*         GradeAntiderivative[result,optimal] returns*)
```

```
(*  "F" if the result fails to integrate an expression that*)
(*      is integrable*)
(*  "C" if result involves higher level functions than necessary*)
(*  "B" if result is more than twice the size of the optimal*)
(*      antiderivative*)
(*  "A" if result can be considered optimal*)


GradeAntiderivative[result_,optimal_] := Module[{expnResult,expnOptimal,leafCountResult,leafC
    expnResult  = ExpnType[result];
    expnOptimal = ExpnType[optimal];
    leafCountResult = LeafCount[result];
    leafCountOptimal = LeafCount[optimal];

    (*Print["expnResult=",expnResult," expnOptimal=",expnOptimal];*)
    If[expnResult<=expnOptimal,
        If[Not[FreeQ[result,Complex]], (*result contains complex*)
            If[Not[FreeQ[optimal,Complex]], (*optimal contains complex*)
                If[leafCountResult<=2*leafCountOptimal,
                    finalresult={"A"," "}
                    ,(*ELSE*)
                    finalresult={"B","Both result and optimal contain complex but leaf count
                ]
            ,(*ELSE*)
                finalresult={"C","Result contains complex when optimal does not."}
            ]
        ,(*ELSE*)(*result does not contains complex*)
            If[leafCountResult<=2*leafCountOptimal,
               finalresult={"A"," "}
            ,(*ELSE*)
               finalresult={"B","Leaf count is larger than twice the leaf count of optimal. $
              ]
        ]
    ,(*ELSE*) (*expnResult>expnOptimal*)
        If[FreeQ[result,Integrate] && FreeQ[result,Int],
            finalresult={"C","Result contains higher order function than in optimal. Order "<
            ,
            finalresult={"F","Contains unresolved integral."}
        ]
    ];

    finalresult
]
```

```
(* ::Text:: *)
(*The following summarizes the type number assigned an *)
(*expression based on the functions it involves*)
(*1 = rational function*)
(*2 = algebraic function*)
(*3 = elementary function*)
(*4 = special function*)
(*5 = hyperpergeometric function*)
(*6 = appell function*)
(*7 = rootsum function*)
(*8 = integrate function*)
(*9 = unknown function*)


ExpnType[expn_] :=
  If[AtomQ[expn],
    1,
  If[ListQ[expn],
    Max[Map[ExpnType,expn]],
  If[Head[expn]===Power,
    If[IntegerQ[expn[[2]]],
      ExpnType[expn[[1]]],
    If[Head[expn[[2]]]===Rational,
      If[IntegerQ[expn[[1]]] || Head[expn[[1]]]===Rational,
        1,
      Max[ExpnType[expn[[1]]],2]],
    Max[ExpnType[expn[[1]]],ExpnType[expn[[2]]],3]]],
  If[Head[expn]===Plus || Head[expn]===Times,
    Max[ExpnType[First[expn]],ExpnType[Rest[expn]]],
  If[ElementaryFunctionQ[Head[expn]],
    Max[3,ExpnType[expn[[1]]]],
  If[SpecialFunctionQ[Head[expn]],
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],4]],
  If[HypergeometricFunctionQ[Head[expn]],
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],5]],
  If[AppellFunctionQ[Head[expn]],
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],6]],
  If[Head[expn]===RootSum,
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],7]],
  If[Head[expn]===Integrate || Head[expn]===Int,
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],8]],
  9]]]]]]]]]]
```

```
ElementaryFunctionQ[func_] :=
  MemberQ[{
  Exp,Log,
  Sin,Cos,Tan,Cot,Sec,Csc,
  ArcSin,ArcCos,ArcTan,ArcCot,ArcSec,ArcCsc,
  Sinh,Cosh,Tanh,Coth,Sech,Csch,
  ArcSinh,ArcCosh,ArcTanh,ArcCoth,ArcSech,ArcCsch
},func]


SpecialFunctionQ[func_] :=
  MemberQ[{
  Erf, Erfc, Erfi,
  FresnelS, FresnelC,
  ExpIntegralE, ExpIntegralEi, LogIntegral,
  SinIntegral, CosIntegral, SinhIntegral, CoshIntegral,
  Gamma, LogGamma, PolyGamma,
  Zeta, PolyLog, ProductLog,
  EllipticF, EllipticE, EllipticPi
},func]


HypergeometricFunctionQ[func_] :=
  MemberQ[{Hypergeometric1F1,Hypergeometric2F1,HypergeometricPFQ},func]


AppellFunctionQ[func_] :=
  MemberQ[{AppellF1},func]
```

4.1.2 Maple grading function

```
# File: GradeAntiderivative.mpl
# Original version thanks to Albert Rich emailed on 03/21/2017

#Nasser 03/22/2017  Use Maple leaf count instead since buildin
#Nasser 03/23/2017  missing 'ln' for ElementaryFunctionQ added
#Nasser 03/24/2017  corrected the check for complex result
#Nasser 10/27/2017  check for leafsize and do not call ExpnType()
#                   if leaf size is "too large". Set at 500,000
```

```
#Nasser 12/22/2019  Added debug flag, added 'dilog' to special functions
#                   see problem 156, file Apostol_Problems
#Nasser 4/07/2022   add second output which gives reason for the grade

GradeAntiderivative := proc(result,optimal)
local leaf_count_result,
        leaf_count_optimal,
        ExpnType_result,
        ExpnType_optimal,
        debug:=false;

        leaf_count_result:=leafcount(result);
        #do NOT call ExpnType() if leaf size is too large. Recursion problem
        if leaf_count_result > 500000 then
            return "B","result has leaf size over 500,000. Avoiding possible recursion issues
        fi;

        leaf_count_optimal := leafcount(optimal);
        ExpnType_result  := ExpnType(result);
        ExpnType_optimal := ExpnType(optimal);

        if debug then
            print("ExpnType_result",ExpnType_result," ExpnType_optimal=",ExpnType_optimal);
        fi;

# If result and optimal are mathematical expressions,
#  GradeAntiderivative[result,optimal] returns
#   "F" if the result fails to integrate an expression that
#       is integrable
#   "C" if result involves higher level functions than necessary
#   "B" if result is more than twice the size of the optimal
#       antiderivative
#   "A" if result can be considered optimal

    #This check below actually is not needed, since I only
    #call this grading only for passed integrals. i.e. I check
    #for "F" before calling this. But no harm of keeping it here.
    #just in case.


    if not type(result,freeof('int')) then
        return "F","Result contains unresolved integral";
    fi;
```

```
if ExpnType_result<=ExpnType_optimal then
    if debug then
         print("ExpnType_result<=ExpnType_optimal");
    fi;
    if is_contains_complex(result) then
        if is_contains_complex(optimal) then
            if debug then
                    print("both result and optimal complex");
            fi;
            if leaf_count_result<=2*leaf_count_optimal then
                return "A"," ";
            else
               return "B",cat("Both result and optimal contain complex but leaf count of
                               convert(leaf_count_result,string)," vs. $2 (",
                               convert(leaf_count_optimal,string)," ) = ",convert(2*leaf_
            end if
        else #result contains complex but optimal is not
            if debug then
                    print("result contains complex but optimal is not");
            fi;
            return "C","Result contains complex when optimal does not.";
        fi;
    else # result do not contain complex
         # this assumes optimal do not as well. No check is needed here.
        if debug then
               print("result do not contain complex, this assumes optimal do not as well"
        fi;
        if leaf_count_result<=2*leaf_count_optimal then
            if debug then
                print("leaf_count_result<=2*leaf_count_optimal");
            fi;
            return "A"," ";
        else
            if debug then
                print("leaf_count_result>2*leaf_count_optimal");
            fi;
            return "B",cat("Leaf count of result is larger than twice the leaf count of o
                               convert(leaf_count_result,string),"$ vs. $2(",
                               convert(leaf_count_optimal,string),")=",convert(2*leaf_cou
        fi;
    fi;
```

```
    else #ExpnType(result) > ExpnType(optimal)
        if debug then
            print("ExpnType(result) > ExpnType(optimal)");
        fi;
        return "C",cat("Result contains higher order function than in optimal. Order ",
                       convert(ExpnType_result,string)," vs. order ",
                       convert(ExpnType_optimal,string),".");
    fi;

end proc:

#
# is_contains_complex(result)
# takes expressions and returns true if it contains "I" else false
#
#Nasser 032417
is_contains_complex:= proc(expression)
  return (has(expression,I));
end proc:

# The following summarizes the type number assigned an expression
# based on the functions it involves
# 1 = rational function
# 2 = algebraic function
# 3 = elementary function
# 4 = special function
# 5 = hyperpergeometric function
# 6 = appell function
# 7 = rootsum function
# 8 = integrate function
# 9 = unknown function

ExpnType := proc(expn)
  if type(expn,'atomic') then
    1
  elif type(expn,'list') then
    apply(max,map(ExpnType,expn))
  elif type(expn,'sqrt') then
    if type(op(1,expn),'rational') then
       1
    else
       max(2,ExpnType(op(1,expn)))
    end if
```

```
  elif type(expn,'`^`') then
    if type(op(2,expn),'integer') then
      ExpnType(op(1,expn))
    elif type(op(2,expn),'rational') then
      if type(op(1,expn),'rational') then
         1
      else
         max(2,ExpnType(op(1,expn)))
      end if
    else
         max(3,ExpnType(op(1,expn)),ExpnType(op(2,expn)))
    end if
  elif type(expn,'`+`') or type(expn,'`*`') then
    max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
  elif ElementaryFunctionQ(op(0,expn)) then
    max(3,ExpnType(op(1,expn)))
  elif SpecialFunctionQ(op(0,expn)) then
    max(4,apply(max,map(ExpnType,[op(expn)])))
  elif HypergeometricFunctionQ(op(0,expn)) then
    max(5,apply(max,map(ExpnType,[op(expn)])))
  elif AppellFunctionQ(op(0,expn)) then
    max(6,apply(max,map(ExpnType,[op(expn)])))
  elif op(0,expn)='int' then
    max(8,apply(max,map(ExpnType,[op(expn)]))) else
  9
  end if
end proc:


ElementaryFunctionQ := proc(func)
  member(func,[
        exp,log,ln,
        sin,cos,tan,cot,sec,csc,
        arcsin,arccos,arctan,arccot,arcsec,arccsc,
        sinh,cosh,tanh,coth,sech,csch,
        arcsinh,arccosh,arctanh,arccoth,arcsech,arccsch])
end proc:

SpecialFunctionQ := proc(func)
  member(func,[
        erf,erfc,erfi,
        FresnelS,FresnelC,
        Ei,Ei,Li,Si,Ci,Shi,Chi,
```

```
        GAMMA,lnGAMMA,Psi,Zeta,polylog,dilog,LambertW,
        EllipticF,EllipticE,EllipticPi])
end proc:

HypergeometricFunctionQ := proc(func)
  member(func,[Hypergeometric1F1,hypergeom,HypergeometricPFQ])
end proc:

AppellFunctionQ := proc(func)
  member(func,[AppellF1])
end proc:

# u is a sum or product.  rest(u) returns all but the
# first term or factor of u.
rest := proc(u) local v;
  if nops(u)=2 then
     op(2,u)
  else
     apply(op(0,u),op(2..nops(u),u))
  end if
end proc:

#leafcount(u) returns the number of nodes in u.
#Nasser 3/23/17  Replaced by build-in leafCount from package in Maple
leafcount := proc(u)
   MmaTranslator[Mma][LeafCount](u);
end proc:
```

4.1.3 Sympy grading function

```
#Dec 24, 2019. Nasser M. Abbasi:
#              Port of original Maple grading function by
#              Albert Rich to use with Sympy/Python
#Dec 27, 2019  Nasser. Added `RootSum`. See problem 177, Timofeev file
#              added 'exp_polar'
from sympy import *

def leaf_count(expr):
   #sympy do not have leaf count function. This is approximation
   return round(1.7*count_ops(expr))

def is_sqrt(expr):
```

```
    if isinstance(expr,Pow):
        if expr.args[1] == Rational(1,2):
            return True
        else:
            return False
    else:
        return False

def is_elementary_function(func):
    return func in [exp,log,ln,sin,cos,tan,cot,sec,csc,
            asin,acos,atan,acot,asec,acsc,sinh,cosh,tanh,coth,sech,csch,
            asinh,acosh,atanh,acoth,asech,acsch
        ]

def is_special_function(func):
    return func in [ erf,erfc,erfi,
             fresnels,fresnelc,Ei,Ei,Li,Si,Ci,Shi,Chi,
             gamma,loggamma,digamma,zeta,polylog,LambertW,
             elliptic_f,elliptic_e,elliptic_pi,exp_polar
         ]

def is_hypergeometric_function(func):
    return func in [hyper]

def is_appell_function(func):
    return func in [appellf1]

def is_atom(expn):
    try:
        if expn.isAtom or isinstance(expn,int) or isinstance(expn,float):
          return True
        else:
          return False

    except AttributeError as error:
        return False

def expnType(expn):
    debug=False
    if debug:
        print("expn=",expn,"type(expn)=",type(expn))

    if is_atom(expn):
```

```
        return 1
    elif isinstance(expn,list):
        return max(map(expnType, expn))   #apply(max,map(ExpnType,expn))
    elif  is_sqrt(expn):
        if isinstance(expn.args[0],Rational): #type(op(1,expn),'rational')
            return 1
        else:
            return max(2,expnType(expn.args[0]))  #max(2,ExpnType(op(1,expn)))
    elif isinstance(expn,Pow):   #type(expn,'`^`')
        if isinstance(expn.args[1],Integer):  #type(op(2,expn),'integer')
            return expnType(expn.args[0])   #ExpnType(op(1,expn))
        elif isinstance(expn.args[1],Rational):  #type(op(2,expn),'rational')
            if isinstance(expn.args[0],Rational): #type(op(1,expn),'rational')
                return 1
            else:
                return max(2,expnType(expn.args[0]))  #max(2,ExpnType(op(1,expn)))
        else:
            return max(3,expnType(expn.args[0]),expnType(expn.args[1])) #max(3,ExpnType(op(1,expn)),ExpnT
    elif isinstance(expn,Add) or isinstance(expn,Mul): #type(expn,'`+`') or type(expn,'`*`')
        m1 = expnType(expn.args[0])
        m2 = expnType(list(expn.args[1:]))
        return max(m1,m2)  #max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
    elif is_elementary_function(expn.func):  #ElementaryFunctionQ(op(0,expn))
        return max(3,expnType(expn.args[0]))  #max(3,ExpnType(op(1,expn)))
    elif is_special_function(expn.func): #SpecialFunctionQ(op(0,expn))
        m1 = max(map(expnType, list(expn.args)))
        return max(4,m1)   #max(4,apply(max,map(ExpnType,[op(expn)])))
    elif is_hypergeometric_function(expn.func): #HypergeometricFunctionQ(op(0,expn))
        m1 = max(map(expnType, list(expn.args)))
        return max(5,m1)   #max(5,apply(max,map(ExpnType,[op(expn)])))
    elif is_appell_function(expn.func):
        m1 = max(map(expnType, list(expn.args)))
        return max(6,m1)   #max(5,apply(max,map(ExpnType,[op(expn)])))
    elif isinstance(expn,RootSum):
        m1 = max(map(expnType, list(expn.args))) #Apply[Max,Append[Map[ExpnType,Apply[List,expn]],7]],
        return max(7,m1)
    elif str(expn).find("Integral") != -1:
        m1 = max(map(expnType, list(expn.args)))
        return max(8,m1)   #max(5,apply(max,map(ExpnType,[op(expn)])))
    else:
        return 9

#main function
```

```
def grade_antiderivative(result,optimal):

    #print ("Enter grade_antiderivative for sagemath")
    #print("Enter grade_antiderivative, result=",result," optimal=",optimal)

    leaf_count_result  = leaf_count(result)
    leaf_count_optimal = leaf_count(optimal)

    #print("leaf_count_result=",leaf_count_result)
    #print("leaf_count_optimal=",leaf_count_optimal)

    expnType_result  = expnType(result)
    expnType_optimal = expnType(optimal)

    if str(result).find("Integral") != -1:
        grade = "F"
        grade_annotation =""
    else:
        if expnType_result <= expnType_optimal:
            if result.has(I):
                if optimal.has(I): #both result and optimal complex
                    if leaf_count_result <= 2*leaf_count_optimal:
                        grade = "A"
                        grade_annotation =""
                    else:
                        grade = "B"
                        grade_annotation ="Both result and optimal contain complex but leaf count of result is large
                else: #result contains complex but optimal is not
                    grade = "C"
                    grade_annotation ="Result contains complex when optimal does not."
            else: # result do not contain complex, this assumes optimal do not as well
                if leaf_count_result <= 2*leaf_count_optimal:
                    grade = "A"
                    grade_annotation =""
                else:
                    grade = "B"
                    grade_annotation ="Leaf count of result is larger than twice the leaf count of optimal. "+str(le
        else:
            grade = "C"
            grade_annotation ="Result contains higher order function than in optimal. Order "+str(ExpnType_


    #print("Before returning. grade=",grade, " grade_annotation=",grade_annotation)
```

```
    return grade, grade_annotation
```

4.1.4 SageMath grading function

```
#Dec 24, 2019. Nasser: Ported original Maple grading function by
#              Albert Rich to use with Sagemath. This is used to
#              grade Fricas, Giac and Maxima results.
#Dec 24, 2019. Nasser: Added 'exp_integral_e' and 'sng', 'sin_integral'
#              'arctan2','floor','abs','log_integral'
#June 4, 2022  Made default grade_annotation "none" instead of "" due
#              issue later when reading the file.
#July 14, 2022. Added ellipticF. This is until they fix sagemath, then remove it.

from sage.all import *
from sage.symbolic.operators import add_vararg, mul_vararg

debug=False;

def tree_size(expr):
    r"""
    Return the tree size of this expression.
    """
    #print("Enter tree_size, expr is ",expr)

    if expr not in SR:
        # deal with lists, tuples, vectors
        return 1 + sum(tree_size(a) for a in expr)
    expr = SR(expr)
    x, aa = expr.operator(), expr.operands()
    if x is None:
        return 1
    else:
        return 1 + sum(tree_size(a) for a in aa)

def is_sqrt(expr):
    if expr.operator() == operator.pow:   #isinstance(expr,Pow):
        if expr.operands()[1]==1/2: #expr.args[1] == Rational(1,2):
            if debug: print ("expr is sqrt")
            return True
        else:
```

```
            return False
    else:
        return False

def is_elementary_function(func):
    #debug=False
    m = func.name() in ['exp','log','ln',
            'sin','cos','tan','cot','sec','csc',
            'arcsin','arccos','arctan','arccot','arcsec','arccsc',
            'sinh','cosh','tanh','coth','sech','csch',
            'arcsinh','arccosh','arctanh','arccoth','arcsech','arccsch','sgn',
        'arctan2','floor','abs'
        ]
    if debug:
        if m:
            print ("func ", func , " is elementary_function")
        else:
            print ("func ", func , " is NOT elementary_function")


    return m

def is_special_function(func):
    #debug=False
    if debug:
        print ("type(func)=", type(func))

    m= func.name() in ['erf','erfc','erfi','fresnel_sin','fresnel_cos','Ei',
          'Ei','Li','Si','sin_integral','Ci','cos_integral','Shi','sinh_integral'
          'Chi','cosh_integral','gamma','log_gamma','psi,zeta',
          'polylog','lambert_w','elliptic_f','elliptic_e','ellipticF',
          'elliptic_pi','exp_integral_e','log_integral']

    if debug:
        print ("m=",m)
        if m:
            print ("func ", func ," is special_function")
        else:
            print ("func ", func ," is NOT special_function")


    return m
```

```
def is_hypergeometric_function(func):
   return func.name() in ['hypergeometric','hypergeometric_M','hypergeometric_U']

def is_appell_function(func):
   return func.name() in ['hypergeometric']   #[appellf1] can't find this in sagemath

def is_atom(expn):

   #debug=False
   if debug:
       print ("Enter is_atom, expn=",expn)

   if not hasattr(expn, 'parent'):
      return False


   #thanks to answer at https://ask.sagemath.org/question/49179/what-is-sagemath-equivalent-to-atomic-
   try:
      if expn.parent() is SR:
         return expn.operator() is None
      if expn.parent() in (ZZ, QQ, AA, QQbar):
         return expn in expn.parent() # Should always return True
      if hasattr(expn.parent(),"base_ring") and hasattr(expn.parent(),"gens"):
         return expn in expn.parent().base_ring() or expn in expn.parent().gens()

      return False

   except AttributeError as error:
      print("Exception,AttributeError in  is_atom")
      print ("cought exception" , type(error).__name__ )
      return False


def expnType(expn):

   if debug:
      print (">>>>>Enter expnType, expn=", expn)
      print (">>>>>is_atom(expn)=", is_atom(expn))

   if is_atom(expn):
      return 1
   elif type(expn)==list:   #isinstance(expn,list):
```

```
        return max(map(expnType, expn))    #apply(max,map(ExpnType,expn))
    elif is_sqrt(expn):
        if type(expn.operands()[0])==Rational: #type(isinstance(expn.args[0],Rational):
            return 1
        else:
            return max(2,expnType(expn.operands()[0]))  #max(2,expnType(expn.args[0]))
    elif expn.operator() == operator.pow:   #isinstance(expn,Pow)
        if type(expn.operands()[1])==Integer:  #isinstance(expn.args[1],Integer)
            return expnType(expn.operands()[0])   #expnType(expn.args[0])
        elif type(expn.operands()[1])==Rational:  #isinstance(expn.args[1],Rational)
            if type(expn.operands()[0])==Rational: #isinstance(expn.args[0],Rational)
                return 1
            else:
                return max(2,expnType(expn.operands()[0]))  #max(2,expnType(expn.args[0]))
        else:
            return max(3,expnType(expn.operands()[0]),expnType(expn.operands()[1])) #max(3,expnType(expn.
    elif expn.operator() == add_vararg or expn.operator() == mul_vararg: #isinstance(expn,Add) or isinst
        m1 = expnType(expn.operands()[0]) #expnType(expn.args[0])
        m2 = expnType(expn.operands()[1:]) #expnType(list(expn.args[1:]))
        return max(m1,m2)  #max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
    elif is_elementary_function(expn.operator()):  #is_elementary_function(expn.func)
        return max(3,expnType(expn.operands()[0]))
    elif is_special_function(expn.operator()): #is_special_function(expn.func)
        m1 = max(map(expnType, expn.operands()))      #max(map(expnType, list(expn.args)))
        return max(4,m1)   #max(4,m1)
    elif is_hypergeometric_function(expn.operator()): #is_hypergeometric_function(expn.func)
        m1 = max(map(expnType, expn.operands()))      #max(map(expnType, list(expn.args)))
        return max(5,m1)   #max(5,m1)
    elif is_appell_function(expn.operator()):
        m1 = max(map(expnType, expn.operands()))      #max(map(expnType, list(expn.args)))
        return max(6,m1)   #max(6,m1)
    elif str(expn).find("Integral") != -1: #this will never happen, since it
                #is checked before calling the grading function that is passed.
                #but kept it here.
        m1 = max(map(expnType, expn.operands()))      #max(map(expnType, list(expn.args)))
        return max(8,m1)   #max(5,apply(max,map(ExpnType,[op(expn)])))
    else:
        return 9

#main function
def grade_antiderivative(result,optimal):
```

```
    if debug:
        print ("Enter grade_antiderivative for sagemath")
        print("Enter grade_antiderivative, result=",result)
        print("Enter grade_antiderivative, optimal=",optimal)
        print("type(anti)=",type(result))
        print("type(optimal)=",type(optimal))

    leaf_count_result  = tree_size(result) #leaf_count(result)
    leaf_count_optimal = tree_size(optimal) #leaf_count(optimal)

    #if debug: print ("leaf_count_result=", leaf_count_result, "leaf_count_optimal=",leaf_count_optimal)


    expnType_result  = expnType(result)
    expnType_optimal = expnType(optimal)

    if debug: print ("expnType_result=", expnType_result, "expnType_optimal=",expnType_optimal)

    if expnType_result <= expnType_optimal:
        if result.has(I):
            if optimal.has(I): #both result and optimal complex
                if leaf_count_result <= 2*leaf_count_optimal:
                    grade = "A"
                    grade_annotation =" "
                else:
                    grade = "B"
                    grade_annotation ="Both result and optimal contain complex but leaf count of result is larger t
            else: #result contains complex but optimal is not
                grade = "C"
                grade_annotation ="Result contains complex when optimal does not."
        else: # result do not contain complex, this assumes optimal do not as well
            if leaf_count_result <= 2*leaf_count_optimal:
                grade = "A"
                grade_annotation =" "
            else:
                grade = "B"
                grade_annotation ="Leaf count of result is larger than twice the leaf count of optimal. "+str(leaf_
    else:
        grade = "C"
        grade_annotation ="Result contains higher order function than in optimal. Order "+str(expnType_resu


    print("Before returning. grade=",grade, " grade_annotation=",grade_annotation)
```

```
    return grade, grade_annotation
```